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Equivariant Alperin-Robinson's Conjecture reduces to almost-simple k^*-groups†

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Abstract: In a recent paper, Gabriel Navarro and Pham Huu Tiep show that the so-called Alperin Weight Conjecture can be verified via the Classification of the Finite Simple Groups, provided any simple group fulfills a very precise list of conditions. Our purpose here is to show that the *equivariant* refinement of the Alperin's Conjecture for blocks formulated by Geoffrey Robinson in the eighties can be reduced to checking the *same* statement on any central k^*-extension of any finite almost-simple group, or of any finite simple group up to verifying an "almost necessary" condition. In an Appendix we develop some old arguments that we need in the proof.

1. Introduction

1.1. In a recent paper [3], Gabriel Navarro and Pham Huu Tiep show that the so-called Alperin Weight Conjecture can be verified *via* the *Classification of the Finite Simple Groups*, provided any simple group fulfills a very precise list of conditions that they consider easier to check than ours, firstly stated in [6, Theorem 16.45] and significantly weakened in [8, Theorem 1.6]††. As a matter of fact, our reduction result concerns *Alperin's Conjecture for blocks* in an *equivariant* formulation which goes back to Geoffrey Robinson in the eighties (it appears in his joint work [11] with Reiner Staszewski).

1.2. Actually, in the introduction of [6] — from I29 to I37 — we consider a refinement of Alperin-Robinson's Conjecture for blocks; but, only in [8] we really show that its verification can be reduced to check that the *same* refinement holds on the so-called *almost-simple k^*-groups* — namely, central k^*-extensions of finite groups H containing a normal simple subgroup S such that H/S is a cyclic p'-group and we have $C_H(S) = \{1\}$. To carry out this checking obviously depends on admitting the *Classification of the Finite Simple Groups*, and our proof of the reduction itself uses the *solvability* of the *outer automorphism group* of a finite simple group (SOFSG), a known fact whose actual proof depends on this classification.

1.3. Our purpose here is, from our results in [6] and [8] that we will recall as far as possible, to show that the Alperin-Robinson's Conjecture for blocks can be reduced to checking the *same* statement on any almost-simple

† We thank Britta Späth for pointing us a mistake in an earlier version of this paper.

†† Gabriel Navarro and Pham Huu Tiep pointed out to us that, when submitting [3], they were not aware of our paper [8], only available in arXiv since April 2010.

k^*-group $\hat{H}$ and moreover, that it may be still reduced to any k^*-central extension of any finite simple group provided we check an "almost necessary" condition (see Proposition 2.14 below) in such an $\hat{H}$. We add an Appendix which actually deals with a more general situation, but provides tools for the proof of our reduction.

1.4. Explicitly, let p be a prime number, k an algebraically closed field of characteristic p, $\mathcal{O}$ a complete discrete valuation ring of characteristic zero admitting k as the *residue* field, and $\mathcal{K}$ the field of fractions of $\mathcal{O}$. Moreover, let $\hat{G}$ be a k^*-central extension of a finite group G — simply called *finite k^*-group* of k^*-quotient G [6, 1.23] — b a block of $\hat{G}$ [6, 1.25] and $\mathcal{G}_k(\hat{G}, b)$ the *scalar extension* from $\mathbb{Z}$ to $\mathcal{O}$ of the *Grothendieck group* of the category of finitely generated $k_*\hat{G}b$-modules [6, 14.3].

1.5. Choose a maximal Brauer $(b, \hat{G})$-pair (P, e); denote by $\mathcal{F}_{(b,\hat{G})}$ the category — called the *Frobenius P-category* of $(b, \hat{G})$ [6, 3.2] — formed by all the subgroups of P and, if Q and R are subgroups of P, by the group homomorphisms $\mathcal{F}_{(b,\hat{G})}(Q, R)$ from R to Q determined by all the elements $x \in G$ fulfilling $(R, g) \subset (Q, f)^x$ where (Q, f) and (R, g) are the corresponding Brauer $(b, \hat{G})$-pairs contained in (P, e); in particular, we set

$$\mathcal{F}_{(b,\hat{G})}(Q) = \mathcal{F}_{(b,\hat{G})}(Q, Q) \cong N_G(Q, f)/C_G(Q) \qquad 1.5.1.$$

Recall that the Brauer $(b, \hat{G})$-pair (Q, f) is called *selfcentralizing* if the image $\bar{f}$ of f in $\bar{C}_{\hat{G}}(Q) = C_{\hat{G}}(Q)/Z(Q)$ is a block of *defect zero* and then we denote by $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$ the full subcategory of $\mathcal{F}_{(b,\hat{G})}$ over the *selfcentralizing* Brauer $(b, \hat{G})$-pairs contained in (P, e).

1.6. Recall that an $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$*-chain* is just a functor $\mathfrak{q} : \Delta_n \to (\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$ from the n-simplex Δ considered as a category with the morphisms given by the order relation; then, the *proper category of* $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$*-chains* — denoted by $\mathfrak{ch}^*\big((\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}\big)$ — is formed by the $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$*-chains* as objects and by the pairs of order-preserving maps and natural isomorphisms of functors as morphisms [6, A2.8]. Denoting by $\mathfrak{Gr}$ the category of finite groups, we actually have a functor [6, Proposition A2.10]

$$\mathfrak{aut}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}} : \mathfrak{ch}^*\big((\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}\big) \longrightarrow \mathfrak{Gr} \qquad 1.6.1$$

mapping the $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$-chain $\mathfrak{q}$ on the stabilizer $\mathcal{F}_{(b,\hat{G})}(\mathfrak{q})$ of $\mathfrak{q}$ in $\mathcal{F}_{(b,\hat{G})}\big(\mathfrak{q}(n)\big)$. Moreover, setting $Q = \mathfrak{q}(n)$ and denoting by f the block of $C_{\hat{G}}(Q)$ such that (P, e) contains (Q, f), it is clear that $N_{\hat{G}}(Q, f)$ acts on the simple k-algebra $k_*\bar{C}_{\hat{G}}(Q)\bar{f}$ and it is well-known that this action determines a central k^*-extension $\hat{\mathcal{F}}_{(b,\hat{G})}(Q)$ of $\mathcal{F}_{(b,\hat{G})}(Q)$ [6, 7.4]; in particular, by restriction we get a central k^*-extension $\hat{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})$ of $\mathcal{F}_{(b,\hat{G})}(\mathfrak{q})$.

1.7. Denoting by k^*-$\mathfrak{Gr}$ the category of k^*-groups with finite k^*-quotient, in [6, Theorem 11.32] we prove the existence of a suitable functor

$$\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat G)})^{\rm sc}} : \mathfrak{ch}^*\big((\mathcal{F}_{(b,\hat G)})^{\rm sc}\big) \longrightarrow k^*\text{-}\mathfrak{Gr} \qquad 1.7.1$$

lifting $\mathfrak{aut}_{(\mathcal{F}_{(b,\hat G)})^{\rm sc}}$ and mapping $\mathfrak{q}$ on $\hat{\mathcal{F}}_{(b,\hat G)}(\mathfrak{q})$; then, still denoting by $\mathcal{G}_k$ the functor mapping any k^*-group with finite k^*-quotient $\hat G$ on the *scalar extension* from $\mathbb{Z}$ to $\mathcal{O}$ of the *Grothendieck group* of the category of finitely generated $k_*\hat G$-modules, and any k^*-group homomorphism on the corresponding restriction, we consider the inverse limit

$$\mathcal{G}_k(\mathcal{F}_{(b,\hat G)}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat G)})^{\rm sc}}) = \lim_{\longleftarrow}\, (\mathcal{G}_k \circ \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat G)})^{\rm sc}}) \qquad 1.7.2,$$

called the *Grothendieck group of* $\mathcal{F}_{(b,\hat G)}$ [6, 14.3.3 and Corollary 14.7]; it follows from [6, I32 and Corollary 14.32] that Alperin's Conjecture for blocks is actually equivalent to the existence of an $\mathcal{O}$-module isomorphism

$$\mathcal{G}_k(\hat G, b) \cong \mathcal{G}_k(\mathcal{F}_{(b,\hat G)}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat G)})^{\rm sc}}) \qquad 1.7.3$$

which actually amounts to saying that both members have the same $\mathcal{O}$-rank.

1.8. Denote by $\mathrm{Out}_{k^*}(\hat G)$ the group of *outer* k^*-automorphisms of $\hat G$ and by $\mathrm{Out}_{k^*}(\hat G)_b$ the stabilizer of b in $\mathrm{Out}_{k^*}(\hat G)$; on the one hand, it is clear that $\mathrm{Out}_{k^*}(\hat G)_b$ acts on $\mathcal{G}_k(\hat G, b)$; on the other hand, an easy *Frattini argument* shows that the stabilizer $\mathrm{Aut}_{k^*}(\hat G)_{(P,e)}$ of (P,e) in $\mathrm{Aut}_{k^*}(\hat G)_b$ *covers* $\mathrm{Out}_{k^*}(\hat G)_b$ and it is clear that it acts on $(\mathcal{F}_{(b,\hat G)})^{\rm sc}$, so that finally $\mathrm{Out}_{k^*}(\hat G)_b$ still acts on the inverse limit $\mathcal{G}_k(\mathcal{F}_{(b,\hat G)}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat G)})^{\rm nc}})$ [6, 16.3 and 16.4]. A stronger question is whether or not in 1.7.3 there exists a $\mathrm{Out}_{k^*}(\hat G)_b$*-stable* isomorphism and in [8, Theorem 1.6] we prove that it suffices to check this statement in the *almost-simple k^*-groups* considered above.

1.9. Here, we are interested in a weaker form of this question, namely in whether or not there exists a $\mathcal{K}\mathrm{Out}_{k^*}(\hat G)_b$-module isomorphism

$$\mathcal{K}\otimes_{\mathcal{O}} \mathcal{G}_k(\hat G, b) \cong \mathcal{K}\otimes_{\mathcal{O}} \mathcal{G}_k(\mathcal{F}_{(b,\hat G)}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat G)})^{\rm sc}}) \qquad 1.9.1;$$

as a matter of fact, it is a *numerical* question since it amounts to saying that the $\mathrm{Out}_{k^*}(\hat G)_b$-characters of both members coincide and note that they are actually *rational* characters. Thus, it makes sense to relate this statement with the old Robinson's *equivariant condition* recalled below. We still need some notation; for any Brauer $(b,\hat G)$-pair (Q,f) contained in (P,e) , the group $\mathcal{F}_Q(Q)$ of inner automorphisms of Q is a normal subgroup of $\mathcal{F}_{(b,\hat G)}(Q)$ and we set (cf. 1.5.1)

$$\tilde{\mathcal{F}}_{(b,\hat G)}(Q) = \mathcal{F}_{(b,\hat G)}(Q)/\mathcal{F}_Q(Q) \cong N_G(Q,f)/Q{\cdot}C_G(Q) \qquad 1.9.2;$$

moreover, if (Q,f) is selfcentralizing then $\mathcal{F}_Q(Q)$ can be identified to a normal p-subgroup of $\hat{\mathcal{F}}_{(b,\hat{G})}(Q)$; then, we also set

$$\hat{\bar{\hat{\mathcal{F}}}}_{(b,\hat{G})}(Q) = \hat{\mathcal{F}}_{(b,\hat{G})}(Q)/\mathcal{F}_Q(Q) \qquad 1.9.3$$

and denote by $o_{(Q,f)}$ the sum of blocks of *defect zero* of $\hat{\bar{\hat{\mathcal{F}}}}_{(b,\hat{G})}(Q)$; note that, since the stabilizer $\mathrm{Aut}_{k^*}(\hat{G})_{(P,e)}$ of (P,e) in $\mathrm{Aut}_{k^*}(\hat{G})_b$ *covers* $\mathrm{Out}_{k^*}(\hat{G})_b$ and acts on $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$, the stabilizer $C_{(Q,f)}$ in a (cyclic) subgroup C of $\mathrm{Out}_{k^*}(\hat{G})_b$ of the G-conjugacy class of (Q,f) acts naturally on $\mathcal{G}_k\big(\hat{\bar{\hat{\mathcal{F}}}}_{(b,\hat{G})}(Q), o_{(Q,f)}\big)$.

1.10. Following Robinson, let us consider the following *equivariant condition*:

(E) *For any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{G})_b$ we have*

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{G},b)^C\big) = \sum_{(Q,f)} \mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\bar{\hat{\mathcal{F}}}}_{(b,\hat{G})}(Q), o_{(Q,f)}\big)^{C_{(Q,f)}}\Big) \qquad 1.10.1$$

where (Q,f) runs over a set of representatives contained in (P,e) for the set of C-orbits of G-conjugacy classes of selfcentralizing Brauer $(b,\hat{G})$-pairs and, for such a (Q,f) , we denote by $C_{(Q,f)}$ the stabilizer in C of the G-conjugacy class of (Q,f) .

We are ready to state our first main result.

Theorem 1.11. *Assume* (SOSFG) *and that any block c of any k^*-extension $\hat{H}$ of any finite group H , containing a finite simple group S such that H/S is a cyclic p'-group and that we have $C_H(S) = \{1\}$, fulfills the equivariant condition* (E). *Then, any block b of any k^*-extension $\hat{G}$ of any finite group G fulfills the equivariant condition* (E) *and, in particular, we have a $\mathcal{K}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism*

$$\mathcal{K}\otimes_{\mathcal{O}} \mathcal{G}_k(\hat{G},b) \cong \mathcal{K}\otimes_{\mathcal{O}} \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}}) \qquad 1.11.1.$$

2. The obstruction

2.1. In order to explain the obstruction to get a better reduction, let $\hat{S}$ be a k^*-group of non-abelian simple k^*-quotient S and d a block of $\hat{S}$ which fulfill condition (E); choose a maximal Brauer $(d,\hat{S})$-pair (P,e) and denote by $\mathcal{Q}$ a set of representatives contained in (P,e) for the set of S-conjugacy classes of selfcentralizing Brauer $(d,\hat{S})$-pairs; once again since the stabilizer $\mathrm{Aut}_{k^*}(\hat{S})_{(P,e)}$ of (P,e) in $\mathrm{Aut}_{k^*}(\hat{S})_d$ *covers* $\mathrm{Out}_{k^*}(\hat{S})_d$ and acts on $(\mathcal{F}_{(d,\hat{S})})^{\mathrm{sc}}$, the group $\mathrm{Out}_{k^*}(\hat{S})_d$ acts on the family $\{\mathcal{G}_k\big(\hat{\bar{\hat{\mathcal{F}}}}_{(d,\hat{S})}(Q), o_{(Q,f)}\big)\}_{(Q,f)\in\mathcal{Q}}$ and

the direct sum of this family becomes an $\mathcal{O}\mathrm{Out}_{k^*}(\hat S)_d$-module. Then, since both $\mathcal{K}\mathrm{Out}_{k^*}(\hat S)_d$-modules

$$\mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k(\hat S,d)\quad\text{and}\quad\bigoplus_{(Q,f)\in\mathcal{Q}}\mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(d,\hat S)}(Q),o_{(Q,f)}\big)\qquad 2.1.1$$

actually come from $\mathbb{Q}\mathrm{Out}_{k^*}(\hat S)_d$-modules, equalities 1.10.1 amount to saying that these $\mathrm{Out}_{k^*}(\hat S)_d$-representations have the same character and therefore that we have a $\mathcal{K}\mathrm{Out}_{k^*}(\hat S)_d$-module isomorphism

$$\mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k(\hat S,d)\cong\bigoplus_{(Q,f)\in\mathcal{Q}}\mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(d,\hat S)}(Q),o_{(Q,f))}\big)\qquad 2.1.2.$$

2.2. Let $\hat H$ be a k^*-group of finite k^*-quotient H in such a way that $\hat H$ contains and normalizes $\hat S$, and c a block of $\hat H$ such that $cd\neq 0$; denoting by $\hat H_d$ the stabilizer of d in $\hat H$, *Fong's reduction* can be written as follows [7, Propositions 3.2 and 3.5]

$$k_*\hat Hc=\mathrm{Ind}_{\hat H_d}^{\hat H}\big(k_*\hat H_d(cd)\big)\qquad 2.2.1$$

and we know that cd is a block of $\hat H_d$; hence, for our purposes, we may assume that $\hat H$ fixes d and thus that we actually have $cd=c$. As above, we assume that $A=H/S$ is a cyclic p'-group and that we have $C_H(S)=\{1\}$.

2.3. On the other hand, it is clear that $C_{\hat H}(P,e)$ acts on the k^*-group $\hat{\bar{\mathcal{F}}}_{(d,\hat S)}(P)$ acting trivially on $\tilde{\mathcal{F}}_{(d,\hat S)}(P)$; let us denote by $K_{\hat H}(P,e)$ the kernel of this action and set

$$\hat L=\hat S{\cdot}K_{\hat H}(P,e)\quad\text{and}\quad D=\hat L/\hat S\qquad 2.3.1;$$

it follows from [8, Proposition 3.8] that c is still a block of $\hat L$ and from [8, Theorem 3.10] that the *source P-interior algebras* of $(d,\hat S)$ and $(c,\hat L)$ are isomorphic; in particular, we have [4, Propositions 6.12 and 14.6]

$$\hat{\bar{\mathcal{F}}}_{(d,\hat S)}(P)=\hat{\bar{\mathcal{F}}}_{(c,\hat L)}(P)\qquad 2.3.2.$$

2.4. More precisely, it follows from [6, Lemma 15.16] that the block e of $C_{\hat S}(P)$ splits into a family $\{e_\varphi\}_{\varphi\in\mathrm{Hom}(D,k^*)}$ of blocks of

$$C_{\hat L}(P,e)=K_{\hat H}(P,e)\qquad 2.4.1$$

and then any (P,e_φ) clearly becomes a maximal Brauer $(d_\varphi,\hat L)$-pair for a suitable block d_φ of $\hat L$; since, by the very definition of $K_{\hat H}(P,e)$, the group $\tilde{\mathcal{F}}_{(d,\hat S)}(P)$ acts trivially on this k^*-group, the number of blocks d_φ coincides with $|\mathrm{Hom}(D,k^*)|$ and then, a simple argument on the dimensions proves that

$$k_*\hat Sd\cong k_*\hat Ld_\varphi\qquad 2.4.2$$

for any $\varphi \in \mathrm{Hom}(D, k^*)$. Moreover, from 2.3 above, we have $d_\varphi = c$ for some choice of φ; set $e_c = e_\varphi$ and note that it follows from equality 2.3.2 above and from [8, Corollary 3.12] that we have

$$\mathcal{F}_{(d,\hat{S})} = \mathcal{F}_{(c,\hat{L})} \quad \text{and} \quad \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(d,\hat{S})})^{\mathrm{sc}}} = \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(c,\hat{L})})^{\mathrm{sc}}} \qquad 2.4.3.$$

Hence, for our purposes, we may replace the pair $(d, \hat{S})$ by the pair $(c, \hat{L})$; in particular, denoting by $\mathcal{R}$ a set of representatives contained in (P, e_c) for the set of L-conjugacy classes of selfcentralizing Brauer $(c, \hat{L})$-pairs, from isomorphism 2.1.2 we get a $\mathcal{K}\mathrm{Out}_{k^*}(\hat{L})_c$-module isomorphism

$$\mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(\hat{L}, c) \cong \bigoplus_{(R,g)\in\mathcal{R}} \mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R), o_{(R,g)}\big) \qquad 2.4.4$$

since $\mathrm{Aut}_{k^*}(\hat{L})$ clearly stabilizes $\hat{S}$.

2.5. Set $B = N_{\hat{H}}(P, e_c)/N_{\hat{L}}(P, e_c) \cong \hat{H}/\hat{L}$; isomorphism 2.4.4 is obviously a $\mathcal{K}B$-isomorphism and therefore, since the group B is cyclic, the respective B-stable $\mathcal{K}$-bases

$$\mathrm{Irr}_k(\hat{L}, c) \quad \text{and} \quad \bigsqcup_{(R,g)\in\mathcal{R}} \mathrm{Irr}_k\big(\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R), o_{(R,g)}\big) \qquad 2.5.1$$

become *isomorphic B-sets*. That is to say, an irreducible Brauer character θ of $\hat{L}$ in the block c determines a selfcentralizing Brauer $(c, \hat{L})$-pair (R, g) in $\mathcal{R}$ and a *projective* irreducible Brauer character θ^* of $\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R)$, in such a way that the stabilizer B_θ of θ in B coincides with the stabilizer of the pair formed by $(R, g) \in \mathcal{R}$ and θ^* .

2.6. On the one hand, note that c is also a block of the stabilizer $\hat{H}_\theta$ of $\theta \in \mathrm{Irr}_k(c, \hat{L})$ in $\hat{H}$; denote by $\mathcal{G}_k(\hat{H}_\theta \,|\, \theta)$ the direct summand of $\mathcal{G}_k(\hat{H}_\theta, c)$ generated by the classes of the simple $k_*\hat{H}_\theta$-modules whose restriction to $\hat{L}$ involves θ; then, it follows from the so-called *Clifford theory* that we have a canonical isomorphism

$$\mathcal{G}_k(\hat{H}_\theta \,|\, \theta) \cong \mathcal{G}_k(\widehat{B}^{\theta}_\theta) \qquad 2.6.1$$

for the central k^*-extension $\widehat{B}^{\theta}_\theta$ of B_θ defined in 2.7.3 below. Moreover, recall that we have

$$\mathcal{G}_k(\hat{H}, c) = \bigoplus_{\theta\in\Theta} \mathrm{Ind}_{\hat{H}_\theta}^{\hat{H}}\big(\mathcal{G}_k(\hat{H}_\theta \,|\, \theta)\big) \qquad 2.6.2$$

where Θ is a set of representatives for the set of H-orbits of $\mathrm{Irr}_k(\hat{L}, c)$. Consequently, since any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{H})_c$ acts on $\mathrm{Irr}_k(\hat{L})$, we have

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{H}, c)^C\big) = \sum_{\theta\in\Theta} \mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\widehat{B}^{\theta}_\theta)^{C_\theta}\big) \qquad 2.6.3$$

where, for any $\theta \in \Theta$, C_θ denotes the stabilizer in C of the H-orbit of θ .

2.7. More explicitly, denoting by V_θ a $k_*\hat{L}$-module affording θ and by $\rho_\theta : \hat{L} \to GL_k(V_\theta)$ the corresponding k^*-group homomorphism, the action of H_θ on $\hat{L}$ determines a group homomorphism $\bar{\rho}_\theta : H_\theta \to PGL_k(V_\theta)$ and we can consider the *pull-back*

$$\begin{array}{ccc} H_\theta & \stackrel{\bar{\rho}_\theta}{\longrightarrow} & PGL_k(V_\theta) \\ \uparrow & & \uparrow \pi_\theta \\ \widehat{H}_\theta^{\theta} & \longrightarrow & GL_k(V_\theta) \end{array} \qquad 2.7.1$$

where $\widehat{H}_\theta^{\theta}$ is the k^*-group formed by the pairs $(x, f) \in H_\theta \times GL_k(V_\theta)$ such that $\bar{\rho}_\theta(x) = \pi_\theta(f)$. Moreover, since the composition of $\bar{\rho}_\theta$ with the map $\hat{L} \to H_\theta$ extends $\pi_\theta \circ \rho_\theta$, we have an injective canonical k^*-group homomorphism $\hat{L} \to \widehat{H}_\theta^{\theta}$, so that we get an injective canonical k^*-group homomorphism

$$L \times k^* \cong \hat{L} * (\hat{L})^\circ \longrightarrow \hat{H}_\theta * (\widehat{H}_\theta^{\theta})^\circ \qquad 2.7.2$$

and, identifying L with its image which is a normal subgroup of $\hat{H}_\theta * (\widehat{H}_\theta^{\theta})^\circ$, we set

$$\widehat{B}_\theta^{\theta} = \big(\hat{H}_\theta * (\widehat{H}_\theta^{\theta})^\circ\big)/L \qquad 2.7.3.$$

2.8. On the other hand, for any selfcentralizing Brauer $(c, \hat{L})$-pair (R, g) contained in (P, e_c), set $\bar{N}_{\hat{L}}(R, g) = N_{\hat{L}}(R, g)/R$ and denote by $\bar{g}$ the image of g in $k_*\bar{N}_{\hat{L}}(R, g)$; since $\bar{g}$ is a block of *defect zero* of $\bar{C}_{\hat{L}}(R)$, applying again *Fong's reduction* we get [7, Proposition 3.2 and Theorem 3.7]

$$k_*\bar{N}_{\hat{L}}(R, g)\bar{g} \cong k_*\bar{C}_{\hat{L}}(R)\bar{g} \otimes_k k_*\hat{\tilde{\mathcal{F}}}_{(c,\hat{L})}(R) \qquad 2.8.1$$

and clearly $\bar{g} \otimes o_{(R,g)}$ corresponds to the sum of all the blocks of *defect zero* in $k_*\bar{N}_{\hat{L}}(R, g)\bar{g}$. Note that, since the quotient $\bar{N}_{\hat{H}}(R, g)/\bar{N}_{\hat{L}}(R, g)$ is a p'-group, the blocks of *defect zero* of $\bar{N}_{\hat{H}}(R, g)$ and $\bar{N}_{\hat{L}}(R, g)$ mutually correspond [6, Proposition 15.9].

2.9. Moreover, since the quotient $E = C_{\hat{H}}(R, g)/C_{\hat{L}}(R)$ is cyclic, it follows again from [6, Lemma 15.16] that the block g of $C_{\hat{L}}(R)$ splits into a family $\{g_\psi\}_{\psi \in \mathrm{Hom}(E,k^*)}$ of blocks of $C_{\hat{H}}(R, g)$ and, according to [8, 3.7], the group $\tilde{\mathcal{F}}_{(c,\hat{L})}(R)$ acts transitively on this family; as in 2.4 above, a simple argument on the dimensions proves that

$$k_*C_{\hat{L}}(R)g \cong k_*C_{\hat{H}}(R, g)g_\psi \qquad 2.9.1$$

for any $\psi \in \mathrm{Hom}(E, k^*)$; setting $\bar{N}_{\hat{H}}(R, g) = N_{\hat{H}}(R, g)/R$, once again *Fong's reduction* provides the following decomposition [7, Proposition 3.2 and Theorem 3.7]

$$k_*\bar{N}_{\hat{H}}(R, g)\bar{g} \cong \mathrm{Ind}_{\bar{N}_{\hat{H}}(R,g_\psi)}^{\bar{N}_{\hat{H}}(R,g)}\big(k_*\bar{C}_{\hat{H}}(R, g)\bar{g}_\psi \otimes_k k_*\hat{\tilde{\mathcal{F}}}_{(c,\hat{H})}(R)\big) \qquad 2.9.2$$

where, as in 1.9.2 above, we have

$$\tilde{\mathcal{F}}_{(c,\hat{H})}(R) \cong \bar{N}_{\hat{H}}(R, g_\psi)/\bar{C}_{\hat{H}}(R) \qquad 2.9.3.$$

2.10. Furthermore, since $\mathrm{Hom}(E,k^*)$ is a p'-group, p does not divide $|\bar N_{\hat H}(R,g)\colon \bar N_{\hat H}(R,g_\psi)|$ and therefore isomorphism 2.9.2 induces a bijection between the sets of isomorphism classes of *projective simple* $k_*\bar N_{\hat H}(R,g)\bar g$- and $k_*\hat{\bar{\mathcal{F}}}_{(c,\hat H)}(R)$-modules; in other words, $\mathrm{Tr}_{\bar N_{\hat H}(R,g_\psi)}^{\bar N_{\hat H}(R,g)}(\bar g_\psi\otimes o_{(R,g_\psi)})$ corresponds to the sum $n_{(R,g)}$ of all the blocks of *defect zero* of $k_*\bar N_{\hat H}(R,g)\bar g$. But, a maximal Brauer $(c,\hat H)$-pair $(P,e_\circ)$ such that $e_\circ$ appears in the decomposition of e_c (cf. 2.4) contains (R,g_ψ) for a unique choice of $\psi\in\mathrm{Hom}(E_\theta,k^*)$, and we set $g_\psi = g_\circ$; in particular, the family $\mathcal{R}_\circ = \{(R,g_\circ)\}_{(R,g)\in\mathcal{R}}$ is a set of representatives contained in $(P,e_\circ)$ for the set of H-conjugacy classes of selfcentralizing Brauer $(c,\hat H)$-pairs. In conclusion, isomorphism 2.9.2 induces a natural $\mathcal{O}$-isomorphism

$$\mathcal{G}_k\big(\bar N_{\hat H}(R,g),n_{(R,g)}\big)\cong \mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(c,\hat H)}(R),o_{(R,g_\circ)}\big) \qquad 2.10.1.$$

2.11. Now, denote by $\Theta_{(R,g)}$ the subset of elements of Θ (cf. 2.6) determining the Brauer $(c,\hat L)$-pair (R,g) ; any $\theta\in\Theta_{(R,g)}$ also determines a *projective* irreducible character θ^* of $\hat{\bar{\mathcal{F}}}_{(c,\hat L)}(R)$ and then, according to isomorphism 2.8.1, θ^* determines a *projective* irreducible Brauer character ζ_{θ^*} of $k_*\bar N_{\hat L}(R,g)\bar g$. Since $\hat H_\theta$ is also the stabilizer in $\hat H$ of the pair formed by $(R,g)\in\mathcal{R}$ and θ^* (cf. 2.5), we have

$$B_\theta\cong N_{\hat H_\theta}(R,g)/N_{\hat L}(R,g) \qquad 2.11.1$$

and, denoting by $\mathcal{G}_k\big(\bar N_{\hat H_\theta}(R,g)\,|\,\zeta_{\theta^*}\big)$ the direct summand of $\mathcal{G}_k\big(\bar N_{\hat H_\theta}(R,g),\bar g\big)$ generated by the classes of simple $k_*\bar N_{\hat H_\theta}(R,g)$-modules whose restriction to $\bar N_{\hat L}(R,g)$ involves ζ_{θ^*} , it follows again from the *Clifford theory* that we have a canonical isomorphism

$$\mathcal{G}_k\big(\bar N_{\hat H_\theta}(R,g)\,|\,\zeta_{\theta^*}\big)\cong \mathcal{G}_k\big(\widehat{B}_\theta^{\theta^*}\big) \qquad 2.11.2$$

for an analogous central k^*-extension $\widehat{B}_\theta^{\theta^*}$ of B_θ . Moreover, since the blocks of *defect zero* of $\bar N_{\hat H}(R,g)$ and of $\bar N_{\hat L}(R,g)$ mutually correspond, as in 2.6 above we have

$$\mathcal{G}_k\big(\bar N_{\hat H}(R,g),n_{(R,g)}\big)=\bigoplus_{\theta\in\Theta_{(R,g)}}\mathrm{Ind}_{N_{\hat H_\theta}(R,g)}^{N_{\hat H}(R,g)}\Big(\mathcal{G}_k\big(\bar N_{\hat H_\theta}(R,g)\,|\,\zeta_{\theta^*}\big)\Big) \qquad 2.11.3.$$

2.12. Finally, it is clear that $\mathrm{Aut}_{k^*}(\hat H)$ stabilizes $\hat S$ and, consequently, that $\mathrm{Aut}_{k^*}(\hat H)_c$ stabilizes $\hat L$ and acts on $\mathcal{G}_k(\hat L,c)$; more precisely, the stabilizer $\mathrm{Aut}_{k^*}(\hat H)_{(P,e_c)}$ of (P,e_c) in $\mathrm{Aut}_{k^*}(\hat H)_c$ acts on $(\mathcal{F}_{(c,\hat L)})^{\mathrm{sc}}$ and the

$\mathcal{K}\mathrm{Out}_{k^*}(\hat{L})_c$-module isomorphism 2.4.4 restricts to a $\mathcal{K}\big(\mathrm{Aut}_{k^*}(\hat{H})_{(P,e_c)}\big)$-isomorphism; moreover, this isomorphism clearly preserves the *$\mathcal{K}B$-isotypic components* of both members and, since $\mathrm{Aut}_{k^*}(\hat{H})_{(P,e_c)}$ *covers* $\mathrm{Out}_{k^*}(\hat{H})_c$, we still have a $\mathcal{K}\mathrm{Out}_{k^*}(\hat{H})_c$-module isomorphism

$$\big(\mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k(\hat{L},c)\big)^B \cong \Big(\bigoplus_{(R,g)\in\mathcal{R}} \mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R), o_{(R,g)}\big)\Big)^B \qquad 2.12.1;$$

in particular, as in 2.4 above, for any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{H})_c$, the respective C-stable $\mathcal{K}$-bases indexed by the quotient sets

$$\mathrm{Irr}_k(\hat{L},c)/B \quad \text{and} \quad \Big(\bigsqcup_{(R,g)\in\mathcal{R}} \mathrm{Irr}_k\big(\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R), o_{(R,g)}\big)\Big)\Big/B \qquad 2.12.2$$

become *isomorphic C-sets*.

2.13. Consequently, for any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{H})_c$, we can choose a *$B$-set isomorphism*

$$\mathrm{Irr}_k(\hat{L},c) \cong \bigsqcup_{(R,g)\in\mathcal{R}} \mathrm{Irr}_k\big(\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R), o_{(R,g)}\big) \qquad 2.13.1$$

inducing a *C-set isomorphism* between the quotient sets in 2.12.2; in this situation, considering $\theta\in\Theta$ and the corresponding pair formed by $(R,g)\in\mathcal{R}$ and by a *projective* irreducible character θ^* of $\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R)$, the stabilizer in $C_{(R,g)}$ of θ^* coincides with C_θ and we have (cf. 2.10.1 and 2.11.2)

$$\begin{aligned}\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(c,\hat{H})}(R), o_{(R,g_\circ)}\big)^{C_{(R,g_\circ)}} &\cong \mathcal{G}_k\big(\bar{N}_{\hat{H}}(R,g), n_{(R,g)}\big)^{C_{(R,g)}}\\ &\cong \bigoplus_{\theta\in\Theta_{(R,g)}} \mathcal{G}_k\big(\bar{N}_{\hat{H}_\theta}(R,g)\,|\,\zeta_{\theta^*}\big)^{C_\theta}\\ &\cong \bigoplus_{\theta\in\Theta_{(R,g)}} \mathcal{G}_k(\widehat{B}_\theta^{\theta^*})^{C_\theta}\end{aligned} \qquad 2.13.2.$$

At this point, it follows from equalities 2.6.3 and isomorphisms 2.13.2 that a *sufficient* statement to guaranteeing that the block c of $\hat{H}$ fulfillis condition (E) is that, for any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{H})_c$ and any $\theta\in\Theta$, the following equality holds

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\widehat{B}_\theta^{\theta})^{C_\theta}\big) = \mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\widehat{B}_\theta^{\theta^*})^{C_\theta}\big) \qquad 2.13.3.$$

Note that this equality forces that the action of C_θ is trivial on $\mathcal{G}_k(\widehat{B}_\theta^{\theta})$ if and only if it is trivial on $\mathcal{G}_k(\widehat{B}_\theta^{\theta^*})$. We are ready to state our second main result.

Proposition 2.14. *With the notation and the hypothesis above, assume that we have a* $\mathcal{K}\mathrm{Out}_{k^*}(\hat{L})_c$*-module isomorphism*

$$\mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k(\hat{L},c)\cong\bigoplus_{(R,g)\in\mathcal{R}}\mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R),o_R\big) \qquad 2.14.1.$$

Let C *be a cyclic subgroup of* $\mathrm{Out}_{k^*}(\hat{H})_c$ *and* θ *and element in* $\mathrm{Irr}_k(\hat{L},c)$ *. If the actions of* C_θ *on* $\mathcal{G}_k(\hat{H}_\theta\mid\theta)$ *and on* $\mathcal{G}_k\big(\bar{N}_{\hat{H}_\theta}(R,g)\mid\zeta_{\theta^*}\big)$ *have the same kernel then we have*

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\widehat{B}_\theta^{\theta})^{C_\theta}\big)=\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\widehat{B}_\theta^{\theta^*})^{C_\theta}\big) \qquad 2.14.2.$$

3. Proof of the second main result

3.1. Let A be a cyclic p'-group and set $\hat{A}=A\times k^*$; we have an obvious split exact sequence

$$1\longrightarrow\mathrm{Hom}(A,k^*)\longrightarrow\mathrm{Aut}_{k^*}(\hat{A})\longrightarrow\mathrm{Aut}(A)\longrightarrow 1 \qquad 3.1.1$$

and a canonical $\mathcal{O}$-module isomorphism

$$\mathcal{G}_k(\hat{A})\cong\mathcal{O}\mathrm{Hom}_{k^*}(\hat{A},k^*) \qquad 3.1.2;$$

moreover, the action of $\mathrm{Hom}(A,k^*)\subset\mathrm{Aut}_{k^*}(\hat{A})$ on $\mathcal{O}\mathrm{Hom}_{k^*}(\hat{A},k^*)$ through isomorphism 3.1.2 is just defined by the "product"

$$\mathrm{Hom}(A,k^*)\times\mathrm{Hom}_{k^*}(\hat{A},k^*)\longrightarrow\mathrm{Hom}_{k^*}(\hat{A},k^*) \qquad 3.1.3.$$

On the other hand, a subgroup of $\mathrm{Hom}(A,k^*)$ is the image of $\mathrm{Hom}(A/D,k^*)$ for some subgroup D of A and it is easily checked that the restriction induces an $\mathcal{O}$-module isomorphism

$$\mathcal{G}_k(\hat{A})^{\mathrm{Hom}(A/D,k^*)}\cong\mathcal{G}_k(\hat{D}) \qquad 3.1.4.$$

Lemma 3.2. *With the notation above, let* C *and* C' *be subgroups of* $\mathrm{Aut}_{k^*}(\hat{A})$ *having the same order and the same image in* $\mathrm{Aut}(A)$ *. Then, we have*

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{A})^C\big)=\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{A})^{C'}\big) \qquad 3.2.1.$$

Proof: We argue by induction on $|A|$; according to our hypothesis, we have

$$|C\cap\mathrm{Hom}(A,k^*)|=|C'\cap\mathrm{Hom}(A,k^*)| \qquad 3.2.2;$$

since $\mathrm{Hom}(A,k^*)$ is cyclic, we actually get the equality

$$C\cap\mathrm{Hom}(A,k^*)=C'\cap\mathrm{Hom}(A,k^*) \qquad 3.2.3$$

and this intersection is the image of $\mathrm{Hom}(A/D, k^*)$ for some subgroup D of A; moreover, the restriction determines a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Hom}(A/D, k^*) & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Hom}(A, k^*) & \longrightarrow & \mathrm{Aut}_{k^*}(\hat{A}) & \longrightarrow & \mathrm{Aut}(A) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Hom}(D, k^*) & \longrightarrow & \mathrm{Aut}_{k^*}(\hat{D}) & \longrightarrow & \mathrm{Aut}(D) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \qquad 3.2.4$$

where all the horizontal sequences are split since we can choose compatible splittings in the middle and the bottom horizontal sequences.

Then, since C and C' have the same image in $\mathrm{Aut}(A)$, they also have the same image in $\mathrm{Aut}(D)$ and we actually get $C \cap X = C' \cap X$, so that the images $\tilde{C}$ and $\tilde{C}'$ of C and C' in $\mathrm{Aut}_{k^*}(\hat{D})$ still have the same order; moreover, according to 3.1.4 and 3.2.3, we obtain

$$\mathcal{G}_k(\hat{A})^C \cong \mathcal{G}_k(\hat{D})^{\tilde{C}} \quad \text{and} \quad \mathcal{G}_k(\hat{A})^{C'} \cong \mathcal{G}_k(\hat{D})^{\tilde{C}'} \qquad 3.2.5;$$

thus, if $D \neq A$ then it suffices to apply our induction hypothesis.

From now on, we assume that

$$C \cap \mathrm{Hom}(A, k^*) = \{1\} = C' \cap \mathrm{Hom}(A, k^*) \qquad 3.2.6.$$

Let us consider the *residual* Grothendieck group of $\hat{A}$ [6, 15.22]

$$\mathcal{R}\mathcal{G}_k(\hat{A}) = \bigcap_E \mathrm{Ker}\big(\mathrm{Res}^{\hat{A}}_{\hat{E}}\big) \qquad 3.2.7$$

where E runs over the set of proper subgroups of A and, for such an E, we denote by

$$\mathrm{Res}^{\hat{A}}_{\hat{E}} : \mathcal{G}_k(\hat{A}) \longrightarrow \mathcal{G}_k(\hat{E}) \qquad 3.2.8$$

the restriction map; it is easily checked that we have a canonical isomorphism [6, 15.23.4]

$$\mathcal{G}_k(\hat{A}) \cong \bigoplus_E \mathcal{R}\mathcal{G}_k(\hat{E}) \qquad 3.2.9$$

where E runs over the set of subgroups of A. In particular, we get

$$\mathcal{G}_k(\hat{A})^C \cong \bigoplus_E \mathcal{R}\mathcal{G}_k(\hat{E})^C \quad \text{and} \quad \mathcal{G}_k(\hat{A})^{C'} \cong \bigoplus_E \mathcal{R}\mathcal{G}_k(\hat{E})^{C'} \qquad 3.2.10$$

where E runs over the set of subgroups of A.

On the other hand, for any proper subgroup E of A, C and C' have as above the same image in $\mathrm{Aut}(E)$; moreover, the products of C and C' by the image of $\mathrm{Hom}(A/E, k^*)$ in $\mathrm{Aut}_{k^*}(\hat{A})$ have the same order, and their images in $\mathrm{Aut}_{k^*}(\hat{E})$ respectively coincide with the images of C and C'; now, the argument above proves that these images also have the same order. Consequently, it follows again from our induction hypothesis that we already have

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{E})^C\big) = \mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{E})^{C'}\big) \qquad 3.2.11.$$

In particular, according to the corresponding isomorphisms 3.2.10, this forces

$$\sum_F \mathrm{rank}_{\mathcal{O}}\big(\mathcal{R}\mathcal{G}_k(\hat{F})^C\big) = \sum_F \mathrm{rank}_{\mathcal{O}}\big(\mathcal{R}\mathcal{G}_k(\hat{F})^{C'}\big) \qquad 3.2.12$$

where F runs over the set of subgroups of E, and, since equalities hold for any proper subgroup E of A, we finally obtain

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{R}\mathcal{G}_k(\hat{E})^C\big) = \mathrm{rank}_{\mathcal{O}}\big(\mathcal{R}\mathcal{G}_k(\hat{E})^{C'}\big) \qquad 3.2.13.$$

In conclusion, always according to the isomorphisms 3.2.10, it remains to prove that

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{R}\mathcal{G}_k(\hat{A})^C\big) = \mathrm{rank}_{\mathcal{O}}\big(\mathcal{R}\mathcal{G}_k(\hat{A})^{C'}\big) \qquad 3.2.14.$$

The equality $\hat{A} = A \times k^*$ defines an splitting of the exact sequence 3.1.1

$$\mathrm{Aut}_{k^*}(\hat{A}) \cong \mathrm{Hom}(A, k^*) \rtimes \mathrm{Aut}(A) \qquad 3.2.15$$

and $\mathcal{O}$-module isomorphisms

$$\mathcal{G}_k(\hat{A}) \cong \mathcal{G}_k(A) \quad \text{and} \quad \mathcal{R}\mathcal{G}_k(\hat{A}) \cong \mathcal{R}\mathcal{G}_k(A) \qquad 3.2.16;$$

we make the corresponding identifications; note that $\mathcal{G}_k(A)$ has an $\mathcal{O}$-algebra structure and that $\mathcal{R}\mathcal{G}_k(A)$ is an ideal; then, for any $\chi \in \mathcal{G}_k(A)$, let us denote by μ_χ the multiplication by χ in $\mathcal{G}_k(A)$. Moreover, denoting by δ_a the characteristic $\mathcal{K}$-valued function of $a \in A$ and by $A^* \subset A$ the set of *generators* of A, it is quite clear that $\{\delta_a\}_{a\in A^*}$ is a $\mathcal{K}$-basis of $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{R}\mathcal{G}_k(A)$.

In particular, the image $\bar{C}$ of C in $\mathrm{Aut}(A)$ acts on $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{R}\mathcal{G}_k(A)$ acting freely on this basis, and for any $\sigma \in C$ there is $\psi_\sigma \in \mathrm{Hom}(A, k^*)$ such that $\sigma = \psi_\sigma {\cdot} \bar{\sigma}$ where $\bar{\sigma}$ denotes the image of σ in $\mathrm{Aut}(A)$ (cf. 3.2.15), so that for any $a \in A^*$ we get

$$\sigma(\delta_a) = (\mu_{\psi_\sigma} \circ \bar{\sigma})(\delta_a) = \psi_\sigma\big(\bar{\sigma}(a)\big){\cdot}\delta_{\bar{\sigma}(a)} \qquad 3.2.17$$

where we identify the group k^* with its canonical lifting to $\mathcal{K}^*$; thus, if $O \subset A^*$ is a $\bar{C}$-orbit then the ideal $\mathcal{I}_O = \bigoplus_{a\in O} \mathcal{K}{\cdot}\delta_a$ of $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(A)$ is C-stable and, in order to prove equality 3.2.13, it suffices to show that $(\mathcal{I}_O)^C$ has dimension one.

But, for any $\sigma, \tau \in C$ in $\mathrm{Aut}_{k^*}(\hat{A})$ we get

$$\psi_{\tau\sigma}\cdot\overline{\tau\sigma} = \tau\sigma = (\psi_\tau\cdot\bar{\tau})(\psi_\sigma\cdot\bar{\sigma}) = \psi_\tau(\bar{\tau}\cdot\psi_\sigma\cdot\bar{\tau}^{-1})\cdot\overline{\tau\sigma} \qquad 3.2.18$$

which forces $\psi_{\tau\sigma} = \psi_\tau(\bar{\tau}\cdot\psi_\sigma\cdot\bar{\tau}^{-1})$; then, choosing $a \in O$, it is clear that the element

$$\chi = \sum_{\sigma\in C} \sigma(\delta_a) = \sum_{\sigma\in C} (\mu_{\psi_\sigma} \circ \bar{\sigma})(\delta_a) \qquad 3.2.19$$

is invertible in $\mathcal{I}_O$ and, moreover, for any $\tau \in C$, we have

$$\begin{aligned} \bar{\tau}(\chi) &= \sum_{\sigma\in C} \bar{\tau}\big((\mu_{\psi_\sigma} \circ \bar{\sigma})(\delta_a)\big) = \sum_{\sigma\in C} \big((\bar{\tau} \circ \mu_{\psi_\sigma} \circ \bar{\tau}^{-1}) \circ \overline{\tau\sigma}\big)(\delta_a) \\ &= \sum_{\sigma\in C} (\mu_{\psi_\tau^{-1}} \circ \mu_{\psi_{\tau\sigma}} \circ \overline{\tau\sigma})(\delta_a) = \psi_\tau^{-1}\chi \end{aligned} \qquad 3.2.20;$$

consequently, for any $a' \in O$ we get

$$\begin{aligned} (\mu_{\chi^{-1}} \circ \tau \circ \mu_\chi)(\delta_{a'}) &= (\mu_{\chi^{-1}} \circ \mu_{\psi_\tau})\bar{\tau}(\chi\delta_{a'}) = \chi^{-1}\psi_\tau\bar{\tau}(\chi)\bar{\tau}(\delta_{a'}) \\ &= \bar{\tau}(\delta_{a'}) \end{aligned} \qquad 3.2.21;$$

that is to say, the actions of τ and $\bar{\tau}$ on $\mathcal{I}_O$ are conjugate each other; since $(\mathcal{I}_O)^{\bar{C}}$ has cleraly dimension one, we are done.

3.3. We are ready to prove Proposition 2.14, so we assume that isomorphism 2.14.1 holds; let C be a cyclic subgroup of $\mathrm{Out}_{k^*}(\hat{H})_c$ and θ an element in $\mathrm{Irr}_k(\hat{L}, c)$; we already know that C_θ stabilizes a pair formed by $(R, g) \in \mathcal{R}$ and by a *projective* irreducible character θ^* of $\hat{\bar{\mathcal{F}}}_{(c,\hat{L})}(R)$, and therefore it acts on B_θ (cf. 2.11); more precisely, C_θ acts on both k^*-groups $\widehat{B}_\theta^\theta$ and $\widehat{B}_\theta^{\theta^*}$ and these actions lift its action on B_θ . Moreover, since B_θ is a cyclic p'-group, we have k^*-isomorphisms

$$\widehat{B}_\theta^\theta \cong B_\theta \times k^* \cong \widehat{B}_\theta^{\theta^*} \qquad 3.3.1$$

lifting the identity on B_θ ; setting $\hat{B}_\theta = B_\theta \times k^*$, since $\mathrm{Aut}_{k^*}(\hat{B})$ acts faithfully on $\mathcal{G}_k(\hat{B})$, if we assume that the actions of C_θ on $\mathcal{G}_k(\hat{H}_\theta \mid \theta)$ and on $\mathcal{G}_k\big(\bar{N}_{\hat{H}_\theta}(R, g) \,|\, \zeta_{\theta^*}\big)$ have the same kernel, then it follows from isomorphisms 2.6.1 and 2.11.2 that the images of C_θ in $\mathrm{Aut}_{k^*}(\widehat{B}_\theta^\theta)$ and in $\mathrm{Aut}_{k^*}(\widehat{B}_\theta^{\theta^*})$ have the same order. Now, equality 2.14.2 follows from lemma 3.2; we are done.

4. Proof of the first main result

4.1. In [8, Theorem 1.6], for any block b of any k^*-group $\hat{G}$ of finite k^*-quotient, choosing a maximal Brauer $(b, \hat{G})$-pair (P, e) and denoting by $\mathcal{F}_{(b,\hat{G})}$ the Frobenius P-category of $(b, \hat{G})$ (cf. 1.5), we prove that the existence of an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_b$- module isomorphism

$$\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}}) \qquad 4.1.1$$

is equivalent to the existence of such an isomorphism for any block c of any *almost simple* k^*-group $\hat{H}$. Although not explicit, it is easy to check that, in all the steps of the proof, if we only assume our isomorphisms defined over the corresponding $\mathcal{K}$-extensions, we still obtain isomorphisms defined over the $\mathcal{K}$-extensions. Consequently, we may apply the following result to our present situation.

Theorem 4.2. *Assume* (SOSFG). *If for any block c of positive defect of any almost simple k^*-group $\hat{H}$ there is a $\mathcal{K}\mathrm{Out}_{k^*}(\hat{H})_c$-module isomorphism*

$$\mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(\hat{H}, c) \cong \mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(\mathcal{F}_{(c,\hat{H})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(c,\hat{H})})^{\mathrm{sc}}}) \qquad 4.2.1,$$

then for any block b of any k^*-group $\hat{G}$ of finite k^*-quotient there is an $\mathcal{K}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism

$$\mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(\hat{G}, b) \cong \mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}}) \qquad 4.2.2.$$

4.3. With the same notation, it is clear that isomorphism 4.2.2 is equivalent to the equality of the $\mathrm{Out}_{k^*}(\hat{G})_b$-characters of both members which, as in 2.1 above, amounts to saying that for any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{G})_b$ we have

$$\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}})^C\Big) = \mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{G}, b)^C\big) \qquad 4.3.1\,.$$

Consequently, in order to prove that Theorem 4.2 implies Theorem 1.11 it suffices to show that, under the hypothesis in Theorem 1.11, we have (cf. 1.9)

$$\begin{aligned}\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}})^C\Big)&\\ = \sum_{(Q,f)} \mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(b,\hat{G})}(Q), o_{(Q,f)}\big)^{C_{(Q,f)}}\Big)&\end{aligned} \qquad 4.3.2$$

where (Q,f) runs over a set of representatives contained in (P,e) for the set of C-orbits of G-conjugacy classes of selfcentralizing Brauer $(b,\hat{G})$-pairs and, for such a (Q,f), we denote by $C_{(Q,f)}$ the stabilizer of the G-conjugacy class of (Q,f) in C. Indeed, our hypothesis in Theorem 1.11 implies that any block c of any almost-simple k^*-group $\hat{H}$ fulfills equalities 1.10.1; then, equalities 4.3.2 show that the pair $(c,\hat{H})$ also fulfills equalities 4.3.1 and therefore isomorphism 4.2.1 holds. At this point, Theorem 4.2 implies that, for any block b of any k^*-group $\hat{G}$ of finite k^*-quotient, isomorphism 4.2.2 holds and therefore the pair $(b,\hat{G})$ fulfills equalities 4.3.1; finally, this time equalities 4.3.2 show that the pair $(b,\hat{G})$ fulfills equalities 1.10.1.

4.4. Note that, arguing by induction on $|G|$, under the hypothesis of Theorem 1.11 we may assume that for any block c of any k^*-group $\hat H$ such that $|H| < |G|$ and any cyclic subgroup D of $\mathrm{Out}_{k^*}(\hat H)_c$ we have

$$\mathrm{rank}_{\mathcal O}\big(\mathcal G_k(\hat H, c)^D\big) = \sum_{(R,g)} \mathrm{rank}_{\mathcal O}\Big(\mathcal G_k\big(\hat{\tilde{\mathcal F}}_{(c,\hat H)}(R), o_{(R,g)}\big)^{D_{(R,g)}}\Big) \qquad 4.4.1$$

where (R,g) runs over a set of representatives contained in a maximal Brauer $(c,\hat H)$-pair for the set of D-orbits of H-conjugacy classes of selfcentralizing Brauer $(c,\hat H)$-pairs. But, as in 2.10 above, denoting by $n_{(R,g)}$ the sum of all the blocks of *defect zero* of $k_*\bar N_{\hat H}(R,g)\bar g$, it follows again from [7, Proposition 3.2 and Theorem 3.7] that we have a canonical isomorphism

$$\mathcal G_k\big(\bar N_{\hat H}(R,g), n_{(R,g)}\big) \cong \mathcal G_k\big(\hat{\tilde{\mathcal F}}_{(c,\hat H)}(R), o_{(R,g)}\big) \qquad 4.4.2;$$

actually, $\mathrm{Tr}_{\bar N_{\hat H}(R,g)}^{\bar N_{\hat H}(R)}(n_{(R,g)})$ is a sum of blocks of *defect zero* of $\bar N_{\hat H}(R)$ and all the blocks of *defect zero* of $\bar N_{\hat H}(R)$ involved in $\mathrm{Br}_R(c)$ appear in these sums. Moreover, for any selfcentralizing Brauer $(b,\hat G)$-pair (Q,f) we have $n_{(Q,f)} \neq 0$ only if $\mathbb O_p\big(\bar N_{\hat G}(Q,f)\big) = \{1\}$; in this case, it is easily checked that any normal p-subgroup U of $\hat G$ is contained in Q and, setting $\hat{\bar G} = \hat G/U$ and $\bar Q = Q/U$ we clearly have $\bar N_{\hat{\bar G}}(\bar Q) \cong \bar N_{\hat G}(Q)$. Consequently, if $\mathbb O_p(\hat G) \neq \{1\}$ then it is easily checked from the induction hypothesis that the pair $(b,\hat G)$ also fulfills equality 4.4.1

4.5. As a matter of fact, in the sequel it is more convenient to consider the *exterior* quotient $\tilde{\mathcal F}_{(b,\hat G)}$ of $\mathcal F_{(b,\hat G)}$ [6, 1.3] formed by the same objects and by the morphisms $\tilde\varphi : R \to Q$ where $\tilde\varphi$ denotes the Q-conjugacy class of an $\mathcal F_{(b,\hat G)}$-morphism $\varphi : R \to Q$, the composition being induced by the composition in $\mathcal F_{(b,\hat G)}$; similarly, $(\tilde{\mathcal F}_{(b,\hat G)})^{\mathrm{sc}}$ is the *full* subcategory of $\tilde{\mathcal F}_{(b,\hat G)}$ determined by the set of selfcentralizing Brauer $(b,\hat G)$-pairs contained in (P,e). As in 1.6 above, we consider the *proper category of* $(\tilde{\mathcal F}_{(b,\hat G)})^{\mathrm{sc}}$*-chains* $\mathfrak{ch}^*\big((\tilde{\mathcal F}_{(b,\hat G)})^{\mathrm{sc}}\big)$ and we still have the corresponding functor [6, Proposition A2.10]

$$\mathfrak{aut}_{(\tilde{\mathcal F}_{(b,\hat G)})^{\mathrm{sc}}} : \mathfrak{ch}^*\big((\tilde{\mathcal F}_{(b,\hat G)})^{\mathrm{sc}}\big) \longrightarrow \mathfrak{Gr} \qquad 4.5.1.$$

In [6, 14.9] we show that this functor can also be lifted to an essentially unique functor

$$\widehat{\mathfrak{aut}}_{(\tilde{\mathcal F}_{(b,\hat G)})^{\mathrm{sc}}} : \mathfrak{ch}^*\big((\tilde{\mathcal F}_{(b,\hat G)})^{\mathrm{sc}}\big) \longrightarrow k^*\text{-}\mathfrak{Gr} \qquad 4.5.2$$

which composed with the canonical funtor

$$\mathfrak{ch}^*\big((\mathcal F_{(b,\hat G)})^{\mathrm{sc}}\big) \longrightarrow \mathfrak{ch}^*\big((\tilde{\mathcal F}_{(b,\hat G)})^{\mathrm{sc}}\big) \qquad 4.5.3$$

admits a natural map from $\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat G)})^{\rm sc}}$ (cf. 1.7.1); then, for any $(\tilde{\mathcal{F}}_{(b,\hat G)})^{\rm sc}$-chain $\mathfrak{q}:\Delta_n \to (\tilde{\mathcal{F}}_{(b,\hat G)})^{\rm sc}$, we denote by $\hat{\tilde{\mathcal{F}}}_{(b,\hat G)}(\mathfrak{q})$ the image of $\mathfrak{q}$ by the functor in 4.5.2.

4.6. We actually will prove equality 4.3.2 in two steps; on the one hand, we will adapt our arguments in the proof of [6, Corollary 14.32] in order to show that

$$\operatorname{rank}_{\mathcal{O}}\Big(\mathcal{G}_k(\mathcal{F}_{(b,\hat G)},\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat G)})^{\rm sc}})^C\Big) = \sum_{\mathfrak{q}\in\mathfrak{Q}_{(b,\hat G)}} (-1)^{\ell(\mathfrak{q})}\operatorname{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b,\hat G)}(\mathfrak{q})\big)^{C_{\mathfrak{q}}}\Big) \qquad 4.6.1$$

where $\mathfrak{Q}_{(b,\hat G)}$ is a set of representatives, *fully normalized* in $\mathcal{F}_{(b,\hat G)}$ (see A7 below), for the set of $\tilde{\mathcal{F}}_{(b,\hat G)}$-isomorphism classes of *regular* $(\tilde{\mathcal{F}}_{(b,\hat G)})^{\rm sc}$-chains (see A6 below) and, for such a $\mathfrak{q}:\Delta_n \to (\tilde{\mathcal{F}}_{(b,\hat G)})^{\rm sc}$, $C_{\mathfrak{q}}$ denotes the stabilizer in C of the isomorphism class of $\mathfrak{q}$ and we set $\ell(\mathfrak{q}) = n$. On the other hand, from Lemmas A13 and A14 below and our induction hypothesis we will prove that

$$\sum_{\mathfrak{q}\in\mathfrak{Q}_{(b,\hat G)}} (-1)^{\ell(\mathfrak{q})}\operatorname{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b,\hat G)}(\mathfrak{q})\big)^{C_{\mathfrak{q}}}\Big) = \sum_{(Q,f)} \operatorname{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b,\hat G)}(Q),o_{(Q,f)}\big)^{C_{(Q,f)}}\Big) \qquad 4.6.2$$

where (Q,f) runs over the same set of representatives as above (cf. 1.9).

4.7. In the first step, we need some notation from [6, Ch. 14]; for any $h\in\mathbb{N}-p\mathbb{N}$, let us denote by U_h the group of h-th roots of unity in $\mathcal{O}^*$ and by $({}^h\tilde{\mathcal{F}}_{(b,\hat G)})^{\rm sc}$ the category where the objects are the pairs Q^ρ determined by a selfcentralizing Brauer $(b,\hat G)$-pair (Q,f) contained in (P,e) (cf. 1.5) and by an injective group homomorphism $\rho: U_h \to \tilde{\mathcal{F}}_{(b,\hat G)}(Q)$ (cf. 1.9.2), and where the morphisms from another such a pair R^σ to Q^ρ are the $\tilde{\mathcal{F}}_{(b,\hat G)}$-morphisms $\tilde\varphi: R\to Q$ such that, for any $\xi\in U_h$, we have $\sigma(\xi)\circ\tilde\varphi = \tilde\varphi\circ\rho(\xi)$ [6, 14.25]. Similarly, we denote by ${}^{U_h}\aleph$ the category of finite U_h-sets — namely, finite sets endowed with a U_h-action — and by [6, 14.21]

$$\mathcal{F}ct_{U_h} : {}^{U_h}\aleph \longrightarrow \mathcal{O}\text{-}\mathfrak{mod} \qquad 4.7.1$$

the *contravariant* functor mapping any finite U_h-set X on the $\mathcal{O}$-module $\mathcal{F}ct_{U_h}(X,\mathcal{O})$ of the $\mathcal{O}$-valued functions over X preserving the U_h-actions — U_h acting on $\mathcal{O}$ by multiplication. Then, we consider the functor

$$\mathfrak{s}_h : ({}^h\tilde{\mathcal{F}}_{(b,\hat G)})^{\rm sc} \longrightarrow {}^{U_h}\aleph \qquad 4.7.2$$

provided by [6, Proposition 14.28] and denote by ${}^{\mathcal{K}}\mathfrak{n}_h$ the extension to $\mathcal{K}$ of the composed functor $\mathcal{F}ct_{U_h}\circ\mathfrak{s}_h$.

4.8. Now, it follows from [6, Theorem 14.30] that, for any $n \geq 1$, we have

$$\mathbb{H}^n\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big) = \{0\} \tag{4.8.1;}$$

moreover, in [4, A3.17] we consider the *stable cohomology* groups, denoted by $\mathbb{H}^n_*\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big)$, computed from the *n-cocycles* and the *n-cobaundaries* which are "stable" by the obvious isomorphisms and, since we are working over the field $\mathcal{K}$, in [6, Propositions A4.13] we prove that, for any $n \in \mathbb{N}$, we have

$$\mathbb{H}^n_*\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big) \cong \mathbb{H}^n\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big) \tag{4.8.2.}$$

Finally, it is quite clear that the category $({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}$ fulfills the condition [6, A5.1.1] and we can consider the *regular* $({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}$-chains, namely the $({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}$-chains $\mathfrak{q}^\eta : \Delta_n \longrightarrow ({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}$ [6, Proposition 14.27] such that $\mathfrak{q}^\eta(i-1 \bullet i)$ is *not* an isomorphism for any $1 \leq i \leq n$ [6, A5.2]; then, in [6, Proposition A4.7] we show that the groups $\mathbb{H}^n_*\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big)$ can be computed from the *regular* $({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}$-chains, namely that, for any $n \in \mathbb{N}$, we have

$$\mathbb{H}^n_*\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big) \cong \mathbb{H}^n_{\rm r}\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big) \tag{4.8.3.}$$

4.9. In conclusion, for any $n \geq 1$, we have

$$\mathbb{H}^n_{\rm r}\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big) = \{0\} \tag{4.9.1;}$$

that is to say, for any $n \in \mathbb{N}$ setting

$$\mathbb{C}^n_{\rm r} = \prod_{\mathfrak{q}^\eta} \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}ct_{U_h}\Big(\mathfrak{s}_h\big(\mathfrak{q}^\eta(0)\big), \mathcal{O}\Big)^{\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})_\eta} \tag{4.9.2}$$

where $\mathfrak{q}$ runs over a set of representatives for the set of isomorphism classes of *regular* $(\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}$-chains, $\eta : U_h \to \tilde{\mathcal{F}}(\mathfrak{q})$ runs over the set of injective group homomorphisms [6, Proposition 14.27] and $\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})_\eta$ denotes the stabilizer of η in $\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})$, we have a *finite* exact sequence

$$0 \longrightarrow \mathbb{H}^0\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big) \longrightarrow \mathbb{C}^0_{\rm r} \longrightarrow \ldots \longrightarrow \mathbb{C}^n_{\rm r} \longrightarrow \ldots \tag{4.9.3.}$$

But, since we are working over $\mathcal{K}$, for any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{G})_b$ we still have the *finite* exact sequence of C-fixed points

$$0 \longrightarrow \mathbb{H}^0\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\rm sc}, {}^{\mathcal{K}}\mathfrak{n}_h\big)^C \longrightarrow (\mathbb{C}^0_{\rm r})^C \longrightarrow \ldots \longrightarrow (\mathbb{C}^n_{\rm r})^C \longrightarrow \ldots \tag{4.9.4.}$$

Consequently, we still get

$$\begin{aligned}&\dim_{\mathcal{K}}\Big(\mathbb{H}^0\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}},{}^{\mathcal{K}}\mathfrak{n}_h\big)^C\Big)\\&=\sum_{\mathfrak{q}^\eta}(-1)^{\ell(\mathfrak{q}^\eta)}\dim_{\mathcal{K}}\Big(\mathcal{K}\otimes_{\mathcal{O}}\mathcal{F}ct_{U_h}\Big(\mathfrak{s}_h\big(\mathfrak{q}^\eta(0)\big),\mathcal{O}\Big)^{\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})_\eta\rtimes C_{\tilde{\mathfrak{q}}^\eta}}\Big)\end{aligned}\tag{4.9.5}$$

where $\mathfrak{q}^\eta$ runs over a set of representatives for the isomorphism classes of *regular* $({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}}$-chains [6, Proposition 14.27] and $C_{\mathfrak{q}^\eta}$ denotes the stabilizer in C of the isomorphism class of $\mathfrak{q}^\eta$.

4.10. Finally, on the one hand, it follows from [6, 14.28.3] that we have

$$\begin{aligned}&\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})},\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}})^C\Big)\\&\qquad=\sum_{h\in\mathbb{N}-p\mathbb{N}}\dim_{\mathcal{K}}\Big(\mathbb{H}^0\big(({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}},{}^{\mathcal{K}}\mathfrak{n}_h\big)^C\Big)\end{aligned}\tag{4.10.1.}$$

On the other hand, for any *regular* $(\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}}$-chain $\mathfrak{q}$, setting $\hat{F}=\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})$ and $F=\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})$, it follows from [6, 14.15.3] that we have

$$\mathcal{G}_k(\hat{F})^{C_{\mathfrak{q}}}\cong\Big(\bigoplus_{h\in\mathbb{N}-p\mathbb{N}}\ \bigoplus_{\eta\in\mathrm{Mon}(U_h,F)}\mathcal{F}ct_{U_h}\Big((\varpi_{h,\hat{F}})^{-1}(\eta),\mathcal{O}\Big)\Big)^{F\rtimes C_{\mathfrak{q}}}\tag{4.10.2}$$

where $C_{\mathfrak{q}}$ denotes the stabilizer in C of the isomorphism class of $\mathfrak{q}$ and, for any $h\in\mathbb{N}-p\mathbb{N}$, setting $\hat{U}_h=U_h\times k^*$ we respectively denote by

$$\mathrm{Mon}\big(U_h,\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})\big)\quad\text{and}\quad\mathrm{Mon}_{k^*}\big(\hat{U}_h,\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})\big)\tag{4.10.3}$$

the sets of injective group and k^*-group homomorphisms from U_h to $\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})$ and from $\hat{U}_h$ to $\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})$, and by

$$\varpi_{h,\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})}:\mathrm{Mon}_{k^*}\big(\hat{U}_h,\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})\big)\longrightarrow\mathrm{Mon}\big(U_h,\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})\big)\tag{4.10.4}$$

the canonical map.

4.11. But, by the very definition of the functor $\mathfrak{s}_h$ in [6, Proposition 14.28], for any *regular* $({}^h\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}}$-chain $\mathfrak{q}^\eta$ we have

$$\mathfrak{s}_h\big(\mathfrak{q}^\eta(0)\big)=(\varpi_{h,\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q}(0))})^{-1}(\iota_0^{\mathfrak{q}}\circ\eta)\cong(\varpi_{h,\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})})^{-1}(\eta)\tag{4.11.1}$$

where $\iota_0^{\mathfrak{q}} : \tilde{\mathcal{F}}(\mathfrak{q}) \to \tilde{\mathcal{F}}\big(\mathfrak{q}(0)\big)$ is the structural map [6, 14.26]. Consequently, isomorphism 4.10.2 becomes

$$\begin{aligned} &\mathcal{G}_k\big(\hat{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})\big)^{C_{\mathfrak{q}}} \\ &\cong \Big(\bigoplus_{h\in\mathbb{N}-p\mathbb{N}} \ \bigoplus_{\eta\in\mathrm{Mon}(U_h,\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q}))} \mathcal{F}ct_{U_h}\Big(\mathfrak{s}_k\big(\mathfrak{q}^{\eta}(0)\big),\mathcal{O}\Big)\Big)^{\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})\rtimes C_{\mathfrak{q}}} \\ &\cong \bigoplus_{h\in\mathbb{N}-p\mathbb{N}} \ \bigoplus_{\eta\in\mathrm{Mon}(U_h,\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q}))} \mathcal{F}ct_{U_h}\Big(\mathfrak{s}_k\big(\mathfrak{q}^{\eta}(0)\big),\mathcal{O}\Big)^{\tilde{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})_\eta\rtimes C_{\mathfrak{q}^\eta}} \end{aligned} \qquad 4.11.2$$

In conclusion, the sum of all the equalities 4.9.5 when h runs over $\mathbb{N}-p\mathbb{N}$ yields

$$\begin{aligned} &\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})},\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}})^C\Big) \\ &\qquad = \sum_{\mathfrak{q}\in\mathfrak{Q}_{(b,\hat{G})}} (-1)^{\ell(\mathfrak{q})}\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})\big)^{C_{\mathfrak{q}}}\Big) \end{aligned} \qquad 4.11.3,$$

proving equality 4.6.1.

4.12. Before going further, note that in the right-hand member many pairs of terms in this sum cancel each other; more precisely, it easily follows from Lemma A13 below that, in this sum, we can replace the set $\mathfrak{Q}_{(b,\hat{G})}$ by the image in $\mathfrak{ch}^*\big((\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}}\big)$ of a set of representatives for the isomorphism classes in the intersection $\mathfrak{N}^{\mathcal{F}_{(b,\hat{G})}}$ and, similarly, from Lemma A14 below that we still can replace this set by the image in $\mathfrak{ch}^*\big((\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}}\big)$ of a set of representatives for the isomorphism classes in the intersection $\mathfrak{R}^{\mathcal{F}_{(b,\hat{G})}}$; explicitly, we get

$$\begin{aligned} &\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})},\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}})^C\Big) \\ &\qquad = \sum_{\check{\mathfrak{q}}\in\mathfrak{R}^{\mathcal{F}_{(b,\hat{G})}}} (-1)^{\ell(\mathfrak{q})}\mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})\big)^{C_{\mathfrak{q}}}\Big) \end{aligned} \qquad 4.12.1$$

where, for any $(\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}}$-chain $\mathfrak{q}$, we denote by $\check{\mathfrak{q}}$ the isomorphism class of any lifting of $\mathfrak{q}$ to an $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$-chain.

4.13. On the other hand, let $\mathfrak{q} : \Delta_n \to (\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}}$ be a *regular* $(\tilde{\mathcal{F}}_{(b,\hat{G})})^{\mathrm{sc}}$-chain *fully normalized* in $\tilde{\mathcal{F}}_{(b,\hat{G})}$ (see A6 and A7 below where we replace $\mathcal{F}_{(b,\hat{G})}$ by $\tilde{\mathcal{F}}_{(b,\hat{G})}$) and consider the corresponding normalizer $N_{\hat{G}}(\mathfrak{q})$; since the Brauer $(b,\hat{G})$-pair $(Q_{\mathfrak{q}},b_{\mathfrak{q}})$ determined by $\mathfrak{q}(n) = Q_{\mathfrak{q}}$ and by the condition

$(Q_{\mathfrak{q}}, b_{\mathfrak{q}}) \subset (P, e)$ is selfcentralizing (cf. 1.5), $b_{\mathfrak{q}}$ is actually a *nilpotent* block of $C_{\hat{G}}(Q_{\mathfrak{q}})$; moreover, it is well-known and easily checked that $b_{\mathfrak{q}}$ remains a block of $N_{\hat{G}}(\mathfrak{q})$. Then, it follows from [10, Theorem 3.5 and Corollary 3.15] (see also [2, Theorems 1.8 and 1.12]) that there exists a suitable k^*-group $\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})$, containing $N_P(\mathfrak{q})$ and admitting the exact sequence

$$1 \longrightarrow \mathfrak{q}(0) \longrightarrow \hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q}) \longrightarrow \hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q}) \longrightarrow 1 \qquad 4.13.1,$$

such that we have canonical isomorphisms

$$\mathcal{G}_k(N_{\hat{G}}(\mathfrak{q}), b_{\mathfrak{q}}) \cong \mathcal{G}_k\big(\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})\big) \cong \mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b,\hat{G})}(\mathfrak{q})\big) \qquad 4.13.2.$$

4.14. Actually, it is easily checked from the corresponding definitions that the k^*-quotient $L_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})$ of $\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})$ coincides with the $\mathcal{F}_{(b,\hat{G})}$*-localizer* $L_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})$ of $\mathfrak{q}$ [6, Theorem 18.6]; in particular, from [6, Remark 18.7] it is easy to check that we have the equivalence of categories

$$N_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q}) \cong \mathcal{F}_{L_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})} \qquad 4.14.1$$

where $\mathcal{F}_{L_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})}$ denotes the *Frobenius category* associated with the group $L_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})$ [6, 1.8]; note that 1 is the unique block of $\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})$ and that we have

$$\mathcal{F}_{L_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})} = \mathcal{F}_{(1,\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q}))} \qquad 4.14.2.$$

Moreover, it follows from [6, Corollary 3.6] that we also have an equivalence of categories

$$N_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q}) \cong \mathcal{F}_{(b_{\mathfrak{q}},N_{\hat{G}}(\mathfrak{q}))} \qquad 4.14.3,$$

so that the blocks $b_{\mathfrak{q}}$ of $N_{\hat{G}}(\mathfrak{q})$ and 1 of $\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})$ have the same *Frobenius* $N_P(\mathfrak{q})$*-category*.

4.15. More precisely, it follows from [10, Corollary 3.15] that the blocks $b_{\mathfrak{q}}$ of $N_{\hat{G}}(\mathfrak{q})$ and 1 of $\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q})$ are *basic Morita equivalent* [5, 7.1]; in particular, it follows from [5, 7.6] that the corresponding functors

$$\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b_{\mathfrak{q}},N_{\hat{G}}(\mathfrak{q}))})^{\mathrm{sc}}} \quad \text{and} \quad \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(1,\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q}))})^{\mathrm{sc}}} \qquad 4.15.1$$

are isomorphic; hence, denoting by $\mathrm{Out}_{k^*}(\hat{G})_{b,\mathfrak{q}}$ the stabilizer of the G-conjugacy class of $\mathfrak{q}$ in $\mathrm{Out}_{k^*}(\hat{G})_b$, we also have an $\mathcal{O}\mathrm{Out}_{k^*}(\hat{G})_{b,\mathfrak{q}}$-isomorphism

$$\begin{aligned} &\mathcal{G}_k(\mathcal{F}_{(b_{\mathfrak{q}},N_{\hat{G}}(\mathfrak{q}))}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b_{\mathfrak{q}},N_{\hat{G}}(\mathfrak{q}))})^{\mathrm{sc}}}) \\ &\qquad \cong \mathcal{G}_k\big(\mathcal{F}_{(1,\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q}))}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(1,\hat{L}_{\mathcal{F}_{(b,\hat{G})}}(\mathfrak{q}))})^{\mathrm{sc}}}\big) \end{aligned} \qquad 4.15.2.$$

But, by the very definition of *inverse limit*, it is quite clear that we have an $\mathcal{O}\mathrm{Out}_{k^*}\big(\hat L_{\mathcal{F}_{(b,\hat G)}}(\mathfrak{q})\big)$-isomorphism (cf. 1.7.2)

$$\mathcal{G}_k\big(\mathcal{F}_{(1,\hat L_{\mathcal{F}_{(b,\hat G)}}(\mathfrak{q}))},\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(1,\hat L_{\mathcal{F}_{(b,\hat G)}}(\mathfrak{q}))})^{\mathrm{sc}}}\big)\cong \mathcal{G}_k\big(\hat L_{\mathcal{F}_{(b,\hat G)}}(\mathfrak{q})\big) \qquad 4.15.3.$$

Consequently, from isomorphisms 4.13.2 we obtain $\mathcal{O}\mathrm{Out}_{k^*}(\hat G)_{b,\mathfrak{q}}$-isomorphisms

$$\mathcal{G}_k(\mathcal{F}_{(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))},\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))})^{\mathrm{sc}}})\cong \mathcal{G}_k(N_{\hat G}(\mathfrak{q}),b_\mathfrak{q})\cong \mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b,\hat G)}(\mathfrak{q})\big) \qquad 4.15.4.$$

4.16. In particular, if we have $\hat G = N_{\hat G}(\mathfrak{q})$, either $\mathfrak{q}(0)=\{1\}$ which forces $P=\{1\}$, so that b is a block of *defect zero* and equality 4.6.2 is *tautologically* true, or we have $\mathbb{O}_p(\hat G)\neq\{1\}$ and therefore the pair $(b,\hat G)$ fulfills equality 4.4.2 (cf. 4.4); in this case, we get equality 4.6.2 from the left-hand isomorphism in 4.15.4. Otherwise, we have $\hat G\neq N_{\hat G}(\mathfrak{q})$ for any *regular* $(\tilde{\mathcal{F}}_{(b,\hat G)})^{\mathrm{sc}}$- chain $\mathfrak{q}$ *fully normalized* in $\tilde{\mathcal{F}}_{(b,\hat G)}$ and therefore it follows from our induction hypothesis (cf. 4.4) that for any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat G)_b$, denoting by $C_\mathfrak{q}$ the stabilizer in C of the isomorphism class of $\mathfrak{q}$, we get

$$\begin{aligned}\mathrm{rank}_\mathcal{O}&\Big(\mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b,\hat G)}(\mathfrak{q})\big)^{C_\mathfrak{q}}\Big)\\ &=\sum_{(Q,f)\in\mathcal{Q}_\mathfrak{q}}\mathrm{rang}_\mathcal{O}\Big(\mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))}(Q),o_{(Q,f)}\big)^{C_{\mathfrak{q},(Q,f)}}\Big)\end{aligned} \qquad 4.16.1.$$

where $\mathcal{Q}_\mathfrak{q}$ denotes a set of representatives, contained in a maximal Brauer $(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))$-pair, for the set of $N_G(\mathfrak{q})$-conjugacy classes of selfcentralizing Brauer $(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))$-pairs and, for any $(Q,f)\in\mathcal{Q}_\mathfrak{q}$, $C_{\mathfrak{q},(Q,f)}$ denotes the stabilizer in $C_\mathfrak{q}$ of the $N_G(\mathfrak{q})$-conjugacy class of (Q,f); note that we have $o_{(Q,f)}\neq 0$ only if

$$\mathbb{O}_p\big(\hat{\tilde{\mathcal{F}}}_{(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))}(Q)\big)=\{1\} \qquad 4.16.2,$$

so that only if Q is $\mathcal{F}_{(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))}$*-radical* (see A4 below).

4.17 At this point, in order to prove equality 4.6.2, we have to compute the double sum (cf. 4.12.1)

$$\sum_{\check{\mathfrak{q}}\in\mathfrak{R}^{\mathcal{F}_{(b,\hat G)}}}\ \sum_{(R,g)\in\mathcal{R}_\mathfrak{q}}(-1)^{\ell(\mathfrak{q})}\mathrm{rang}_\mathcal{O}\Big(\mathcal{G}_k\big(\hat{\tilde{\mathcal{F}}}_{(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))}(R),o_{(R,g)}\big)^{C_{\mathfrak{q},(R,g)}}\Big) \qquad 4.17.1.$$

where, for any $\check{\mathfrak{q}}\in\mathfrak{R}^{\mathcal{F}_{(b,\hat G)}}$, $\mathcal{R}_\mathfrak{q}$ is the subset of $R\in\mathcal{Q}_\mathfrak{q}$ which are $\mathcal{F}_{(b_\mathfrak{q},N_{\hat G}(\mathfrak{q}))}$-*radical*. But, for such a $\mathfrak{q}:\Delta_n\to(\mathcal{F}_{(b,\hat G)})^{\mathrm{sc}}$, $\mathfrak{q}(n)$ is actually a normal p-subgroup of $N_{\hat G}(\mathfrak{q})$ and thus, according to Lemma A5 below, R contains $\mathfrak{q}(n)$; thus, if $R\cong\mathfrak{q}(0)$ then $n=0$; otherwise, either $R\neq\mathfrak{q}(n)$ and we consider

the *regular* $(\mathcal{F}_{(b,\hat G)})^{^{\rm sc}}$-chain $\mathfrak{q}^\varpi : \Delta_{n+1} \to (\mathcal{F}_{(b,\hat G)})^{^{\rm sc}}$ extending $\mathfrak{q}$ and mapping $n+1$ on R and the Δ_n-morphism $n \bullet n+1$ on the corresponding inclusion map, or $R = \mathfrak{q}(n)$ and we can consider the restriction $\mathfrak{q}^\varpi : \Delta_{n-1} \to (\mathcal{F}_{(b,\hat G)})^{^{\rm sc}}$ of $\mathfrak{q}$. In both cases, R remains an $N_{\mathcal{F}_{(b,\hat G)}}(\mathfrak{q}^\varpi)$-*radical* subgroup of $N_P(\mathfrak{q}^\varpi)$ and we clearly have

$$\begin{aligned}\Big(\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(b_{\mathfrak{q}^\varpi},N_{\hat G}(\mathfrak{q}^\varpi))}(R), o_{(R,g)}\big)\Big)^{C_{\mathfrak{q}^\varpi,(R,g)}}&\\ \cong \Big(\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(b_{\mathfrak{q}},N_{\hat G}(\mathfrak{q}))}(R), o_{(R,g)}\big)\Big)^{C_{\mathfrak{q},(R,g)}}&\end{aligned} \qquad 4.17.2;$$

moreover, we obviously have $(\mathfrak{q}^\varpi)^\varpi = \mathfrak{q}$. In conclusion, since we clearly have $|\ell(\mathfrak{q}^\varpi) - \ell(\mathfrak{q})| = 1$, the double sum 4.17.1 becomes

$$\sum_{(R,g)} \mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(b,\hat G)}(R), o_{(R,g)}\big)^{C_{(R,g)}}\Big) \qquad 4.17.3$$

where (R,g) runs over a set of representatives contained in (P,e) for the set of G-conjugacy classes of selfcentralizing Brauer $(b,\hat G)$-pairs such that R is a fully normalized $\mathcal{F}_{(b,\hat G)}$-*radical* subgroup of P, proving equality 4.6.2. We are done.

Appendix: Radical functions over folded Frobenius categories

A1. The contents of this Appendix rises from [12] where Jacques Théve-naz adopts the old point of view consisting on that, in a finite group G, the word "local" is synonymous of "concerning the family of normalizers of nontrivial p-subgroups". Here we exhibit what seems a more adequate frame-work, involving Frobenius categories. As a matter of fact, our arguments are useful in the proof of our main result above. Let P be a finite p-group and denote by $\mathfrak{iGr}$ the category formed by the finite groups and by the injective group homomorphisms, and by $\mathcal{F}_P$ the subcategory of $\mathfrak{iGr}$ where the objects are all the subgroups of P and the morphisms are the group homomorphisms induced by conjugation by elements of P.

A2. Recall that a *Frobenius P-category* $\mathcal{F}$ is a subcategory of $\mathfrak{iGr}$ con-taining $\mathcal{F}_P$ where the objects are all the subgroups of P and the morphisms fulfill the following three conditions [6, 2.8 and Proposition 2.11]

A2.1 *For any subgroup Q of P the inclusion functor $(\mathcal{F})_Q \to (\mathfrak{iGr})_Q$ is full.*

A2.2 *$\mathcal{F}_P(P)$ is a Sylow p-subgroup of $\mathcal{F}(P)$.*

A2.3 *If Q is a subgroup of P fulfilling $\xi\big(C_P(Q)\big) = C_P\big(\xi(Q)\big)$ for any $\mathcal{F}$-morphism $\xi : Q{\cdot}C_P(Q) \to P$, if $\varphi : Q \to P$ is an $\mathcal{F}$-morphism and if R is a subgroup of $N_P\big(\varphi(Q)\big)$ containing $\varphi(Q)$ such that $\mathcal{F}_P(Q)$ contains the action of $\mathcal{F}_R\big(\varphi(Q)\big)$ over Q via φ, then there is an $\mathcal{F}$-morphism $\zeta : R \to P$ fulfilling $\zeta\big(\varphi(u)\big) = u$ for any $u \in Q$.*

As in [6, 1.2], for any pair of subgroups Q and R of P, we denote by $\mathcal{F}(Q,R)$ the set of $\mathcal{F}$-morphisms from Q to R and set $\mathcal{F}(Q)=\mathcal{F}(Q,Q)$; moreover, recall that, for any category $\mathfrak{C}$ and any $\mathfrak{C}$-object C, $\mathfrak{C}_C$ (or $(\mathfrak{C})_C$ to avoid confusion) denotes the category of "$\mathfrak{C}$-morphisms to C" [6, 1.7].

A3. Given a Frobenius P-category $\mathcal{F}$, a subgroup Q of P and a subgroup K of the group $\mathrm{Aut}(Q)$ of automorphisms of Q, we say that Q is *fully K-normalized* in $\mathcal{F}$ if we have [6, 2.6]

$$\xi\big(N_P^K(Q)\big)=N_P^{{}^\xi K}\big(\xi(Q)\big) \qquad A3.1$$

for any $\mathcal{F}$-morphism $\xi:Q{\cdot}N_P^K(Q)\to P$, where $N_P^K(Q)$ is the converse image of K in $N_P(Q)$ *via* the canonical group homomorphism $N_P(Q)\to\mathrm{Aut}(Q)$ and ${}^\xi K$ is the image of K in $\mathrm{Aut}\big(\xi(Q)\big)$ *via* ξ. Recall that if Q is fully K-normalized in $\mathcal{F}$ then we have a new Frobenius $N_P^K(Q)$-category $N_\mathcal{F}^K(Q)$ where, for any pair of subgroups R and T of $N_P^K(Q)$, $\big(N_\mathcal{F}^K(Q)\big)(R,T)$ is the set of group homomorphisms *from T to R* induced by the $\mathcal{F}$-morphisms $\psi:Q{\cdot}T\to Q{\cdot}R$ which stabilize Q and induce on it an element of K [6, 2.14 and Proposition 2.16].

A4. We say that a subgroup Q of P is *$\mathcal{F}$-selfcentralizing* if we have

$$C_P\big(\varphi(Q)\big)\subset\varphi(Q) \qquad A4.1$$

for any $\varphi\in\mathcal{F}(P,Q)$, and we denote by $\mathcal{F}^{^{\mathrm{sc}}}$ the full subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of P. From the case of the Frobenius P-categories *associated with a block of a finite group*, we know that it only makes sense to consider central k^*-extensions of $\mathcal{F}(Q)$ whenever Q is $\mathcal{F}$-selfcentralizing [6, 7.4]; but, if U is a subgroup of P fully K-normalized in $\mathcal{F}$ for some subgroup K of $\mathrm{Aut}(U)$, a $N_\mathcal{F}^K(U)$-selfcentralizing subgroup of $N_P(Q)$ need not be $\mathcal{F}$-selfcentralizing, which is a handicap when comparing choices of central k^*-extensions in $\mathcal{F}$ and in $N_F^K(U)$. In order to overcome this difficulty, we consider the *$\mathcal{F}$-radical* subgroups of P; we say that a subgroup R of P is *$\mathcal{F}$-radical* if it is $\mathcal{F}$-selfcentralizing and we have

$$\mathbf{O}_p\big(\tilde{\mathcal{F}}(R)\big)=\{1\} \qquad A4.2$$

where $\tilde{\mathcal{F}}(R)=\mathcal{F}(R)/\mathcal{F}_R(R)$ [6, 1.3]; we denote by $\mathcal{F}^{^{\mathrm{rd}}}$ the full subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-radical subgroups of P.

Lemma A5 *Let $\mathcal{F}$ be a Frobenius P-category, U a subgroup of P and K a subgroup of* $\mathrm{Aut}(U)$ *containing* $\mathrm{Int}(U)$. *If U is fully K-normalized in $\mathcal{F}$ then any $N_\mathcal{F}^K(U)$-radical subgroup R of $N_P^K(U)$ contains U and, in particular, it is $\mathcal{F}$-selfcentralizing.*

Proof: It is quite clear that the image of $N_{U\cdot R}(R)$ in $\big(N_\mathcal{F}^K(U)\big)(R)$ is a normal p-subgroup and therefore it is contained in $\mathbf{O}_p\Big(\big(N_\mathcal{F}^K(U)\big)(R)\Big)$, so that $N_{U\cdot R}(R)=R$ which forces $U{\cdot}R=R$.

Moreover, for any $\mathcal{F}$-morphism $\psi : R \to P$, it is clear that $\psi(U)$ is a normal subgroup of $\psi(R){\cdot}C_P\big(\psi(R)\big)$ and therefore, since U is also fully centralized in $\mathcal{F}$ [6, Proposition 2.12], it follows from A2.3 that there is an $\mathcal{F}$-morphism

$$\zeta : \psi(R){\cdot}C_P\big(\psi(R)\big) \longrightarrow P \qquad A5.1$$

fulfilling $\zeta\big(\psi(u)\big) = u$ for any $u \in U$, so that the group homomorphism from R to $N_P^K(U)$ mapping $v \in R$ on $\zeta\big(\psi(v)\big)$ is a $N_{\mathcal{F}}^K(U)$-morphism; in particular, $\zeta\big(\psi(R)\big)$ is also $N_{\mathcal{F}}^K(U)$-selfcentralizing and therefore we get

$$\zeta\Big(C_P\big(\psi(R)\big)\Big) \subset \zeta\big(\psi(R)\big) \qquad A5.2$$

which forces $C_P\big(\psi(R)\big) \subset \psi(R)$. We are done.

A6. Here, we have to deal with $\mathcal{F}^{^{\rm sc}}$*-chains* and *coherent* choices of central k^*-extensions for the $\mathcal{F}^{^{\rm sc}}$-automorphism groups. Recall that we call $\mathcal{F}^{^{\rm sc}}$*-chain* any functor $\mathfrak{q} : \Delta_n \to \mathcal{F}^{^{\rm sc}}$ where the n-simplex Δ_n is considered as a category with the morphisms defined by the order relation [6, A2.2]; let us call n the *length* of $\mathfrak{q}$ and set $n = \ell(\mathfrak{q})$; recall that $\mathfrak{q}$ is *regular* if $\mathfrak{q}(i{-}1 \bullet i)$ is *not* an isomorphism for any $1 \le i \le n$ [6, A5.2]. Then, we consider the category $\mathfrak{ch}^*(\mathcal{F}^{^{\rm sc}})$ where the objects are all the $\mathcal{F}^{^{\rm sc}}$-chains $\mathfrak{q}$ and the morphisms from $\mathfrak{q} : \Delta_n \to \mathcal{F}^{^{\rm sc}}$ to another $\mathcal{F}^{^{\rm sc}}$-chain $\mathfrak{r} : \Delta_m \to \mathcal{F}^{^{\rm sc}}$ are the pairs (ν, δ) formed by an *order-preserving map* or, equivalently, a functor $\delta : \Delta_m \to \Delta_n$ and by a natural isomorphism $\nu : \mathfrak{q} \circ \delta \cong \mathfrak{r}$, the composition being defined by the composition of maps and of natural isomorphisms [6, A2.8].

A7. We say that an $\mathcal{F}^{^{\rm sc}}$*-chain* $\mathfrak{q} : \Delta_n \to \mathcal{F}^{^{\rm sc}}$ is *fully normalized* in $\mathcal{F}$ if $\mathfrak{q}(n)$ is fully normalized in $\mathcal{F}$ and if, moreover, setting $P' = N_P\big(\mathfrak{q}(n)\big)$ and $\mathcal{F}' = N_{\mathcal{F}}\big(\mathfrak{q}(n)\big)$, whenever $n \ge 1$ the $\mathcal{F}'$-chain $\mathfrak{q}' : \Delta_{n-1} \to \mathcal{F}'$ mapping $i \in \Delta_{n-1}$ on the *image* of $\mathfrak{q}(i \bullet n)$, and the Δ_{n-1}-morphisms on the corresponding inclusion maps, is *fully normalized* in $\mathcal{F}'$ [6, 2.18]; note that, by [6, Proposition 2.7], any $\mathcal{F}^{^{\rm sc}}$-chain admits a $\mathfrak{ch}^*(\mathcal{F}^{^{\rm sc}})$-isomorphic $\mathcal{F}$-chain *fully normalized* in $\mathcal{F}$. Moreover, if $\mathfrak{q}$ is fully normalized in $\mathcal{F}$ and $n \ge 1$, we inductively define [6, 2.19]

$$N_P(\mathfrak{q}) = N_{P'}(\mathfrak{q}') \quad \text{and} \quad N_{\mathcal{F}}(\mathfrak{q}) = N_{\mathcal{F}'}(\mathfrak{q}') \qquad A7.1,$$

and it follows from [6, Proposition 2.16] that $N_{\mathcal{F}}(\mathfrak{q})$ is a Frobenius $N_P(\mathfrak{q})$-category; actually, according to [6, Lemma 2.17] and denoting by $\mathcal{F}(\mathfrak{q})$ the image in $\mathcal{F}\big(\mathfrak{q}(n)\big)$ of the group of natural automorphisms of $\mathfrak{q}$, $\mathfrak{q}(n)$ is *fully* $\mathcal{F}(\mathfrak{q})$*-normalized* in $\mathcal{F}$ and we have

$$N_P(\mathfrak{q}) = N_P^{\mathcal{F}(\mathfrak{q})}\big(\mathfrak{q}(n)\big) \quad \text{and} \quad N_{\mathcal{F}}(\mathfrak{q}) = N_{\mathcal{F}}^{\mathcal{F}(\mathfrak{q})}\big(\mathfrak{q}(n)\big) \qquad A7.2.$$

Recall that we have a canonical functor [6, Proposition A2.10]

$$\mathfrak{aut}_{\mathcal{F}^{\mathrm{sc}}} : \mathfrak{ch}^*(\mathcal{F}^{^{\mathrm{sc}}}) \longrightarrow \mathfrak{Gr} \qquad A7.3$$

mapping any $\mathcal{F}^{^{\mathrm{sc}}}$-chain $\mathfrak{q} : \Delta_n \to \mathcal{F}^{^{\mathrm{sc}}}$ on $\mathcal{F}(\mathfrak{q})$.

A8. We define a *folded Frobenius category* as a triple $(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ formed by a finite p-group P, by a Frobenius P-category $\mathcal{F}$ and by the choice of a functor

$$\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}} : \mathfrak{ch}^*(\mathcal{F}^{^{\mathrm{sc}}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \qquad A8.1$$

lifting $\mathfrak{aut}_{\mathcal{F}^{\mathrm{sc}}}$; note that, for any finite k^*-group $\hat{G}$ and any block b of $\hat{G}$, denoting by P a defect p-subgroup of b, Theorem 11.32 in [6] guarantees de existence of a *folded Frobenius category* $(P, \mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}})$. *Mutatis mutandis*, we consider the category $\mathfrak{ch}^*(\mathcal{F}^{^{\mathrm{rd}}})$ and the canonical functor

$$\mathfrak{aut}_{\mathcal{F}^{\mathrm{rd}}} : \mathfrak{ch}^*(\mathcal{F}^{^{\mathrm{rd}}}) \longrightarrow \mathfrak{Gr} \qquad A8.2;$$

then, it follows from Lemma A5 above and from the following result [9, Theorem 2.9] that, for any subgroup U of P fully K-normalized in $\mathcal{F}$ for some subgroup K of $\mathrm{Aut}(U)$, we still get a *folded Frobenius category* $N^K_{(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})}(U)$ formed by the p-group $N^K_P(U)$, the Frobenius $N^K_P(U)$-category $N^K_{\mathcal{F}}(U)$ and the unique functor

$$\widehat{\mathfrak{aut}}_{N^K_{\mathcal{F}}(U)^{\mathrm{sc}}} : \mathfrak{ch}^*\big(N^K_{\mathcal{F}}(U)^{^{\mathrm{sc}}}\big) \longrightarrow k^*\text{-}\mathfrak{Gr} \qquad A8.3$$

extending the restriction of $\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}}$ to $\mathfrak{ch}^*\big(N^K_{\mathcal{F}}(U)^{^{\mathrm{rd}}}\big)$.

Theorem A9. *Any functor* $\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{rd}}}$ *lifting* $\mathfrak{aut}_{\mathcal{F}^{\mathrm{rd}}}$ *to the category* k^*-$\mathfrak{Gr}$ *can be extended to a unique functor lifting* $\mathfrak{aut}_{\mathcal{F}^{\mathrm{sc}}}$.

$$\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}} : \mathfrak{ch}^*(\mathcal{F}^{^{\mathrm{sc}}}) \longrightarrow k^*\text{-}\mathfrak{Gr} \qquad A9.1$$

A10. On the other hand, let us call *k^*-localizer* any k^*-group $\hat{L}$ with finite k^*-quotient L fulfilling $C_L\big(\mathbb{O}_p(L)\big) = Z\big(\mathbb{O}_p(L)\big)$; note that 1 is the unique block of $\hat{L}$. Following Dade, let us call *radical chain* of $\hat{L}$ any subset $\mathfrak{r}$ of p-subgroups of $\hat{L}$ which is *totally ordered* by the inclusion and, for any $R \in \mathfrak{r}$, fulfills $R = \mathbb{O}_p\big(N_{\hat{L}}(\mathfrak{r}^R)\big)$ where $\mathfrak{r}^R$ is the subset of $\mathfrak{r}$ of all the elements contained in R; note that any element of $\mathfrak{r}$ contains $\mathbb{O}_p(L)$ and that $\mathfrak{r}$ can be identified with a *regular* $\mathcal{F}^{^{\mathrm{sc}}}_{(1,\hat{L})}$-chain. Now, a $\mathbb{Q}$-valued function f defined over the set of *isomorphism classes* of folded Frobenius categories is called *radical* whenever there exists a $\mathbb{Q}$-valued function f^* defined over the set of isomorphism classes of *k^*-localizers* such that for any *folded Frobenius category* $(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ we have

$$f(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}}) = \sum_R f^*\big(\hat{L}_{\mathcal{F}}(R)\big) \qquad A10.1$$

where R runs over a set of representatives *fully normalized* in $\mathcal{F}$ for the set of $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-radical subgroups of P and, for such an R, $L_{\mathcal{F}}(R)$ denotes the $\mathcal{F}$-localizer of R [6, Theorem 18.6] and $\hat{L}_{\mathcal{F}}(R)$ is the *k^*-localizer* coming from the *pull-back*

$$\begin{array}{ccc} \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}}(R) & \longrightarrow & \mathcal{F}(R) \\ \uparrow & & \uparrow \\ \hat{L}_{\mathcal{F}}(R) & \longrightarrow & L_{\mathcal{F}}(R) \end{array} \qquad A10.2.$$

Theorem A11. *A $\mathbb{Q}$-valued function f defined over the set of isomorphism classes of folded Frobenius categories is radical if and only if for any folded Frobenius category $(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ we have*

$$f(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}}) = \sum_{\mathfrak{q}} (-1)^{\ell(\mathfrak{q})} f\big(N_P(\mathfrak{q}), N_{\mathcal{F}}(\mathfrak{q}), \widehat{\mathfrak{aut}}_{N_{\mathcal{F}}(\mathfrak{q})^{\mathrm{sc}}}\big) \qquad A11.1$$

where $\mathfrak{q}$ runs over a set of representatives, fully normalized in $\mathcal{F}$, for the set of $\mathcal{F}$-isomorphism classes of regular $\mathcal{F}^{\mathrm{sc}}$-chains. In this case, for any k^*-localizer $\hat{L}$, choosing a Sylow p-subgroup Q of $\hat{L}$ we have

$$f^*(\hat{L}) = \sum_{\mathfrak{r}} (-1)^{\ell(\mathfrak{r})} f(N_Q(\mathfrak{r}), \mathcal{F}_{(1,N_{\hat{L}}(\mathfrak{r}))}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}_{(1,N_{\hat{L}}(\mathfrak{r}))}}) \qquad A11.2$$

where $\mathfrak{r}$ runs over a set of representatives, contained in Q and fully normalized in $\mathcal{F}_{(1,\hat{L})}$, for the set of $\hat{L}$-conjugacy classes of radical chains of $\hat{L}$ such that $\mathfrak{r}(0) = \mathbb{O}_p(\hat{L})$.

Proof: Firstly assume that f fulfills all the equalities A11.1; then, we claim that it suffices to choose the function f^* defined by the equalities A11.2; that is to say, we claim that for any folded Frobenius category $(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ we have

$$\begin{aligned} &f(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}}) \\ &= \sum_R \sum_{\mathfrak{r}} (-1)^{\ell(\mathfrak{r})} f(N_{N_P(R)}(\mathfrak{r}), \mathcal{F}_{(1,N_{\hat{L}_{\mathcal{F}}(R)}(\mathfrak{r}))}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}_{(1,N_{\hat{L}_{\mathcal{F}}(R)}(\mathfrak{r}))}}) \end{aligned} \qquad A11.3$$

where R runs over a set of representatives *fully normalized* in $\mathcal{F}$ for the set of $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-radical subgroups of P and, for such an R, $\mathfrak{r}$ runs over a set of representatives, contained in $N_P(R)$ and fully normalized in $\mathcal{F}_{(1,\hat{L}_{\mathcal{F}}(R))}$, for the set of $\hat{L}_{\mathcal{F}}(R)$-conjugacy classes of radical chains of $\hat{L}_{\mathcal{F}}(R)$ such that $\mathfrak{r}(0) = R$.

But, it is quite clear that such an $\mathfrak{r}$ can be considered as a *regular* $\mathcal{F}^{\mathrm{sc}}$-chain which is also fully normalized in $\mathcal{F}$, and that two of them $\mathfrak{r}$ and $\mathfrak{r}'$ fulfilling $\mathfrak{r}(0) = \mathfrak{r}'(0)$ are $\hat{L}_{\mathcal{F}}\big(\mathfrak{r}(0)\big)$-conjugate if and only if they are $\mathcal{F}$-isomorphic; moreover, we clearly have [6, Theorem 18.6]

$$N_{N_P(R)}(\mathfrak{r}) = N_P(\mathfrak{r}) \quad \text{and} \quad \mathcal{F}_{(1,N_{\hat{L}_{\mathcal{F}}(R)}(\mathfrak{r}))} \cong N_{\mathcal{F}}(\mathfrak{r}) \qquad A11.4.$$

That is to say, the sum in the right-hand member of equality A11.3 contains all the terms of the sum in the right-hand member of equality A11.1 and therefore, in order to prove our claim, it suffices to show that the sum of the remaining terms is equal to zero. Hence, our claim follows from Lemmas A13 and A14 below since the set of isomorphism classes of the remaining terms coincides with the set $\mathfrak{N}_0^{\mathcal{F}} - \mathfrak{R}^{\mathcal{F}}$ defined below.

Conversely, assume that f is radical; first of all, we prove that f determines f^* ; it suffices to show that a $\mathbb{Q}$-valued function g defined over the set of isomorphism classes of k^**-localizers* $\hat{L}$ vanish if any *folded Frobenius category* $(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ fulfills

$$0 = \sum_R g\big(\hat{L}_{\mathcal{F}}(R)\big) \qquad A11.5$$

where R runs over a set of representatives *fully normalized* in $\mathcal{F}$ for the set of $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-radical subgroups of P ; we argue by induction on $|L|$.

We choose a Sylow p-subgroup Q of L and apply equality A11.5 to the folded Frobenius category $(Q, \mathcal{F}_{(1,\hat{L})}, \widehat{\mathfrak{aut}}_{\mathcal{F}_{(1,\hat{L})}})$, so that we have

$$0 = \sum_R g\big(\hat{L}_{\mathcal{F}_{(1,\hat{L})}}(R)\big) \qquad A11.6$$

where R runs over a set of representatives *fully normalized* in $\mathcal{F}_{(1,\hat{L})}$ for the set of $\mathcal{F}_{(1,\hat{L})}$-isomorphism classes of $\mathcal{F}_{(1,\hat{L})}$-radical subgroups of Q ; but, it follows from Lemma A5 that such an R contains $\mathbb{O}_p(L)$; moreover, it follows from [6, Corollary 3.6 and Theorem 18.6] that we have

$$\hat{L}_{\mathcal{F}_{(1,\hat{L})}}(R) \cong N_{\hat{L}}(R) \qquad A11.7$$

and it is clear that $\mathbb{O}_p(L)$ is an $\mathcal{F}_{(1,\hat{L})}$-radical subgroup of Q ; since $R \neq \mathbb{O}_p(L)$ forces $|N_L(R)| < |L|$, the induction hypothesis implies that all the terms but one vanish in the right hand member of equality A11.6, so that we also obtain $g(\hat{L}) = 0$.

Finally, we have to prove that equality A11.1 holds; let $(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ be a folded Frobenius category; then, for any $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{q} : \Delta_n \to \mathcal{F}^{\mathrm{sc}}$ fully normalized in $\mathcal{F}$, we have the folded Frobenius category $\big(N_P(\mathfrak{q}), N_{\mathcal{F}}(\mathfrak{q}), \widehat{\mathfrak{aut}}_{N_{\mathcal{F}}(\mathfrak{q})^{\mathrm{sc}}}\big)$ and therefore we still have

$$f\big(N_P(\mathfrak{q}), N_{\mathcal{F}}(\mathfrak{q}), \widehat{\mathfrak{aut}}_{N_{\mathcal{F}}(\mathfrak{q})^{\mathrm{sc}}}\big) = \sum_R f^*\big(\hat{L}_{N_{\mathcal{F}}(\mathfrak{q})}(R)\big) \qquad A11.8$$

where R runs over a set of representatives *fully normalized* in $N_{\mathcal{F}}(\mathfrak{q})$ for the set of $N_{\mathcal{F}}(\mathfrak{q})$-isomorphism classes of $N_{\mathcal{F}}(\mathfrak{q})$-radical subgroups of $N_P(\mathfrak{q})$;

consequently, we get

$$\sum_{\mathfrak{q}}(-1)^{\ell(\mathfrak{q})} f\big(N_P(\mathfrak{q}), N_{\mathcal{F}}(\mathfrak{q}), \widehat{\mathfrak{aut}}_{N_{\mathcal{F}}(\mathfrak{q})^{\mathrm{sc}}}\big) = \sum_{\mathfrak{q}}(-1)^{\ell(\mathfrak{q})} \sum_{R} f^*\big(\hat{L}_{N_{\mathcal{F}}(\mathfrak{q})}(R)\big) \qquad A11.9$$

where $\mathfrak{q}$ runs over a set of representatives, fully normalized in $\mathcal{F}$, for the set of $\mathcal{F}$-isomorphism classes of *regular* $\mathcal{F}^{^{\mathrm{sc}}}$-chains and, for such a $\mathfrak{q}$, R runs over a set of representatives *fully normalized* in $N_{\mathcal{F}}(\mathfrak{q})$ for the set of $N_{\mathcal{F}}(\mathfrak{q})$-isomorphism classes of $N_{\mathcal{F}}(\mathfrak{q})$-radical subgroups of $N_P(\mathfrak{q})$.

Once again, it follows from Lemma A13 below that it suffices to consider the sum whenever $\mathfrak{q}$ belongs to $\mathfrak{N}^{^{\mathcal{F}}}$; moreover, we may assume that $\mathfrak{q}(i{-}1)$ is contained in $\mathfrak{q}(i)$ and that $\mathfrak{q}(i{-}1\bullet i)$ is the inclusion map for any $1 \le i \le n$; in this case, since $\mathfrak{q}(i) \subset N_P(\mathfrak{q})$ for any $i \in \Delta_n$, we actually have

$$\mathfrak{q}(i) \subset \mathbb{O}_p\big(\hat{L}_{\mathcal{F}}(\mathfrak{q})\big) \qquad A11.10$$

for any $i \in \Delta_n$; but, it follows from Lemma A5 that R contains $\mathbb{O}_p\big(\hat{L}_{\mathcal{F}}(\mathfrak{q})\big)$; in particular, if $R = \mathfrak{q}(0)$ then $n = 0$; otherwise, either we have $R \neq \mathfrak{q}(n)$ and we consider the *regular* $\mathcal{F}^{^{\mathrm{sc}}}$-chain $\mathfrak{q}^\tau : \Delta_{n+1} \to \mathcal{F}^{^{\mathrm{sc}}}$ extending $\mathfrak{q}$ and mapping $n{+}1$ on R and the Δ_n-morphism $n{\bullet}n{+}1$ on the corresponding inclusion map, or we have $R = \mathfrak{q}(n)$ and we can consider the restriction $\mathfrak{q}^\tau : \Delta_{n-1} \to \mathcal{F}^{^{\mathrm{sc}}}$ of $\mathfrak{q}$.

In both cases, $\mathfrak{q}^\tau$ belongs to $\mathfrak{N}^{^{\mathcal{F}}}$ and we have $R \neq \mathfrak{q}^\tau(0)$; moreover, up to replacing R and $\mathfrak{q}$ by their image through a suitable $\mathcal{F}^{^{\mathrm{sc}}}$-morphism $R \to P$, we may assume that $\mathfrak{q}^\tau$ is fully normalized in $\mathcal{F}$ and then it is easily checked that we get a k^*-isomorphism

$$\hat{L}_{N_{\mathcal{F}}(\mathfrak{q})}(R) \cong \hat{L}_{N_{\mathcal{F}}(\mathfrak{q}^\tau)}(R) \qquad A11.11;$$

consequently, since we have $(\mathfrak{q}^\tau)^\tau = \mathfrak{q}$, in the sum of the right-hand member in A11.9 only remain the terms where $\mathfrak{q}(0) = R$ and $n = 0$, and in this case we have $\hat{L}_{N_{\mathcal{F}}(\mathfrak{q})}(R) = \hat{L}_{\mathcal{F}}(R)$. In conclusion, we obtain

$$\sum_{\mathfrak{q}}(-1)^{\ell(\mathfrak{q})} f\big(N_P(\mathfrak{q}), N_{\mathcal{F}}(\mathfrak{q}), \widehat{\mathfrak{aut}}_{N_{\mathcal{F}}(\mathfrak{q})^{\mathrm{sc}}}\big) = \sum_{R} f^*\big(\hat{L}_{\mathcal{F}}(R)\big) \qquad A11.12$$

where $\mathfrak{q}$ and R respectively run over sets of representatives fully normalized in $\mathcal{F}$ for the set of $\mathcal{F}$-isomorphism classes of *regular* $\mathcal{F}^{^{\mathrm{sc}}}$-chains and for the set of $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-radical subgroups of P, so that the right-hand member coincides with $f(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$. We are done.

A12. Let $(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ be a folded Frobenius category. For any $i\in\mathbb{N}$, we denote by $\mathfrak{N}_i^{\mathcal{F}}$ the set of $\mathcal{F}$-isomorphisms classes of *regular* $\mathcal{F}^{\mathrm{sc}}$-chains $\mathfrak{q}:\Delta_n\to\mathcal{F}^{\mathrm{sc}}$ such that, for any $j\in\Delta_n$, if $j<i$ then the image of $\mathfrak{q}(j)$ in $\mathfrak{q}(n)$ is *normal*; moreover, setting $\mathfrak{N}^{\mathcal{F}}=\cap_{i\in\mathbb{N}}\mathfrak{N}_i^{\mathcal{F}}$, we denote by $\mathfrak{R}_i^{\mathcal{F}}$ the set of $\mathcal{F}$-isomorphisms classes in $\mathfrak{N}^{\mathcal{F}}$ of $\mathcal{F}^{\mathrm{sc}}$-chains $\mathfrak{q}:\Delta_n\to\mathcal{F}^{\mathrm{sc}}$ such that, for any $j\in\Delta_n$, if $j<i$ then $\mathfrak{q}(j)$ is $N_{\mathcal{F}}(\mathfrak{q}^j)$-radical where $\mathfrak{q}^j:\Delta_j\to\mathcal{F}^{\mathrm{sc}}$ is the restriction of $\mathfrak{q}$, up to replacing $\mathfrak{q}^j$ by an $\mathcal{F}$-isomorphic $\mathcal{F}^{\mathrm{sc}}$-chain fully normalized in $\mathcal{F}$; finally, we set $\mathfrak{R}^{\mathcal{F}}=\cap_{i\in\mathbb{N}}\mathfrak{R}_i^{\mathcal{F}}$. It is clear that the stabilizer $\mathrm{Aut}(P)_{\mathcal{F}}$ of $\mathcal{F}$ in $\mathrm{Aut}(P)$ acts on $\mathfrak{N}_i^{\mathcal{F}}$ and on $\mathfrak{R}_i^{\mathcal{F}}$ for any $i\in\mathbb{N}$.

Lemma A13. *With the notation above, for any $i\geq 1$ there is an $\mathrm{Out}(P)_{\mathcal{F}}$-stable involution τ_i of the set $\mathfrak{N}_{i-1}^{\mathcal{F}}-\mathfrak{N}_i^{\mathcal{F}}$ such that, if the isomorphism class $\tilde{\mathfrak{q}}$ of an $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{q}$ fully normalized in $\mathcal{F}$ belongs to $\mathfrak{N}_{i-1}^{\mathcal{F}}-\mathfrak{N}_i^{\mathcal{F}}$, then we have*

$$\begin{gathered}N_{(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})}(\mathfrak{q})\cong N_{(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})}(\mathfrak{q}^{\tau_i})\\ \hat{L}_{\mathcal{F}}(\mathfrak{q})\cong\hat{L}_{\mathcal{F}}(\mathfrak{q}^{\tau_i})\quad\text{and}\quad|\ell(\mathfrak{q})-\ell(\mathfrak{q}^{\tau_i})|=1\end{gathered}\qquad A13.1$$

for a choice in $\tau_i(\tilde{\mathfrak{q}})$ of an $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{q}^{\tau_i}$ fully normalized in $\mathcal{F}$.

Proof: We may assume that $\mathfrak{N}_{i-1}^{\mathcal{F}}-\mathfrak{N}_i^{\mathcal{F}}\neq\emptyset$ and let $\mathfrak{q}:\Delta_n\to\mathcal{F}^{\mathrm{sc}}$ be an $\mathcal{F}^{\mathrm{sc}}$-chain fully normalized in $\mathcal{F}$ with its isomorphism class $\tilde{\mathfrak{q}}$ in this set; we consider the minimal $j\in\Delta_n$ such that $i\leq j$ and that the image Q of $\mathfrak{q}(i-1)$ in $\mathfrak{q}(j)$ is not normal; then, $N_{\mathfrak{q}(j)}(Q)$ is a proper subgroup of $\mathfrak{q}(j)$ containing the image of $\mathfrak{q}(j-1)$. If $N_{\mathfrak{q}(j)}(Q)$ coincides with this image, we have $i\neq j-1$ and we consider the $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{q}':\Delta_{n-1}\to\mathcal{F}^{\mathrm{sc}}$ which coincides with $\mathfrak{q}$ over Δ_{j-2} and maps $\ell\geq j-1$ on $\mathfrak{q}(\ell+1)$; otherwise, we consider the $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{q}':\Delta_{n+1}\to\mathcal{F}^{\mathrm{sc}}$ which coincides with $\mathfrak{q}$ over Δ_{j-1} and maps j on $N_{\mathfrak{q}(j)}(Q)$ and $\ell\geq j+1$ on $\mathfrak{q}(\ell-1)$. In both cases, note that the isomorphism class of $\mathfrak{q}'$ still belongs to $\mathfrak{N}_{i-1}^{\mathcal{F}}-\mathfrak{N}_i^{\mathcal{F}}$ and that $j\in\Delta_n$ is also the minimal element such that $i\leq j$ and that the image of $\mathfrak{q}'(i-1)$ in $\mathfrak{q}'(j)$ is not normal.

Let us replace $\mathfrak{q}'$ by an isomorphic $\mathcal{F}^{\mathrm{sc}}$-chain $\mathfrak{q}^{\tau_i}$ fully normalized in $\mathcal{F}$; in both cases, it is easily checked that such an $\mathcal{F}$-isomorphism induces the following $\mathcal{F}$-isomorphism, equivalence of categories and natural isomorphism

$$N_P(\mathfrak{q})\cong N_P(\mathfrak{q}^{\tau_i})\ ,\ N_{\mathcal{F}}(\mathfrak{q})\cong N_{\mathcal{F}}(\mathfrak{q}^{\tau_i})\ \text{and}\ \widehat{\mathfrak{aut}}_{N_{\mathcal{F}}(\mathfrak{q})^{\mathrm{sc}}}\cong\widehat{\mathfrak{aut}}_{N_{\mathcal{F}}(\mathfrak{q}^{\tau_i})^{\mathrm{sc}}}\quad A13.2;$$

consequently, according to [6, Theorem 18.6] and the *pull-back* A10.2, we get

$$\hat{L}_{\mathcal{F}}(\mathfrak{q})\cong\hat{L}_{\mathcal{F}}(\mathfrak{q}^{\tau_i})\qquad A13.3.$$

Thus, it suffices to define τ_i as the map sending the isomorphism class of $\mathfrak{q}$ to the isomorphism class of $\mathfrak{q}^{\tau_i}$. We are done.

Lemma A14. *With the notation above, for any $i \geq 1$ there is an* $\mathrm{Out}(P)_{\mathcal{F}}$*-stable involution ϖ_i of the set $\mathfrak{R}_{i-1}^{\mathcal{F}} - \mathfrak{R}_i^{\mathcal{F}}$ such that, if the isomorphism class $\tilde{\mathfrak{q}}$ of an $\mathcal{F}^{^{\rm sc}}$-chain $\mathfrak{q}$ fully normalized in $\mathcal{F}$ belongs to $\mathfrak{R}_{i-1}^{\mathcal{F}} - \mathfrak{R}_i^{\mathcal{F}}$, then we have*

$$\begin{gathered} N_{(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\rm sc}})}(\mathfrak{q}) \cong N_{(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\rm sc}})}(\mathfrak{q}^{\varpi_i}) \\ \hat{L}_{\mathcal{F}}(\mathfrak{q}) \cong \hat{L}_{\mathcal{F}}(\mathfrak{q}^{\varpi_i}) \quad and \quad |\ell(\mathfrak{q}) - \ell(\mathfrak{q}^{\varpi_i})| = 1 \end{gathered} \qquad A14.1$$

for a choice in $\varpi_i(\tilde{\mathfrak{q}})$ of an $\mathcal{F}^{^{\rm sc}}$-chain $\mathfrak{q}^{\varpi_i}$ fully normalized in $\mathcal{F}$.

Proof: We may assume that $\mathfrak{R}_{i-1}^{\mathcal{F}} - \mathfrak{R}_i^{\mathcal{F}} \neq \emptyset$ and let $\mathfrak{q} : \Delta_n \to \mathcal{F}^{^{\rm sc}}$ be an $\mathcal{F}^{^{\rm sc}}$-chain fully normalized in $\mathcal{F}$ with its isomorphism class $\tilde{\mathfrak{q}}$ in this set; that is to say, since $\mathfrak{R}_{i-1}^{\mathcal{F}} \subset \mathfrak{N}^{\mathcal{F}}$, $\mathfrak{q}(i-1)$ is contained in $N_P(\mathfrak{q}^{i-1})$ and it is not $N_{\mathcal{F}}(\mathfrak{q}^{i-1})$-*radical* (cf. A7); thus, the structural image of $\mathfrak{q}(i-1)$ in $L_{\mathcal{F}}(\mathfrak{q}^{i-1})$ is a proper subgroup of $\mathbb{O}_p\big(L_{\mathcal{F}}(\mathfrak{q}^{i-1})\big)$, and we consider the maximal $j \in \Delta_n$ such that $i-1 \leq j$ and that the structural image Q of $\mathfrak{q}(j)$ in $L_{\mathcal{F}}(\mathfrak{q}^j)$ is a proper subgroup of $R = \mathbb{O}_p\big(L_{\mathcal{F}}(\mathfrak{q}^j)\big)$.

First of all, note that R normalizes the structural image of $\mathfrak{q}^j$ in $L_{\mathcal{F}}(\mathfrak{q}^j)$; moreover, if $j < n$ then the structural image of $\mathfrak{q}(j+1)$ in $L_{\mathcal{F}}(q^{j+1})$ coincides with $\mathbb{O}_P\big(L_{\mathcal{F}}(\mathfrak{q}^{j+1})\big)$ and therefore, since we have [6, 2.13.2 and Proposition 18.16]

$$L_{\mathcal{F}}(\mathfrak{q}^{j+1}) \cong N_{L_{\mathcal{F}}(\mathfrak{q}^j)}(T) \qquad A14.2$$

where T denotes the structural image of $\mathfrak{q}(j+1)$ in $L_{\mathcal{F}}(q^j)$, we still have $N_R(T) \subset T$, so that T contains R; in conclusion, the structural image of $\mathfrak{q}(j+1)$ in $L_{\mathcal{F}}(q^j)$ contains $\mathbb{O}_p\big(L_{\mathcal{F}}(\mathfrak{q}^j)\big)$ which properly contains the structural image of $\mathfrak{q}(j)$. If $j < n$ and $T = R$ then we consider the $\mathcal{F}^{^{\rm sc}}$-chain $\mathfrak{q}' : \Delta_{n-1} \to \mathcal{F}^{^{\rm sc}}$ which coincides with $\mathfrak{q}$ over Δ_j and maps $\ell \geq j+1$ on $\mathfrak{q}(\ell+1)$; otherwise, we consider the $\mathcal{F}^{^{\rm sc}}$-chain $\mathfrak{q}' : \Delta_{n+1} \to \mathcal{F}^{^{\rm sc}}$ which coincides with $\mathfrak{q}$ over Δ_j and maps $j+1$ on $\mathbb{O}_p\big(L_{\mathcal{F}}(\mathfrak{q}^j)\big)$ and $\ell \geq j+1$ on $\mathfrak{q}(\ell-1)$. In both cases, note that the isomorphism class of $\mathfrak{q}'$ still belongs to $\mathfrak{R}_{i-1}^{\mathcal{F}} - \mathfrak{R}_i^{\mathcal{F}}$ and that $j \in \Delta_n$ is also the maximal element such that $i-1 \leq j$ and that the structural image of $\mathfrak{q}'(j)$ in $L_{\mathcal{F}}(\mathfrak{q}'^j)$ is a proper subgroup of $\mathbb{O}_p\big(L_{\mathcal{F}}(\mathfrak{q}'^j)\big)$.

Let us replace $\mathfrak{q}'$ by an isomorphic $\mathcal{F}^{^{\rm sc}}$-chain $\mathfrak{q}^{\varpi_i}$ fully normalized in $\mathcal{F}$; in both cases, it is easily checked that such an $\mathcal{F}$-isomorphism induces the following isomorphism of *folded Frobenius categories*

$$N_{(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\rm sc}})}(\mathfrak{q}) \cong N_{(P,\mathcal{F},\widehat{\mathfrak{aut}}_{\mathcal{F}^{\rm sc}})}(\mathfrak{q}^{\varpi_i}) \qquad A14.3;$$

consequently, according to [6, Theorem 18.6] and the *pull-back* A10.2, we get

$$\hat{L}_{\mathcal{F}}(\mathfrak{q}) \cong \hat{L}_{\mathcal{F}}(\mathfrak{q}^{\varpi_i}) \qquad A14.4.$$

Thus, it suffices to define ϖ_i as the map sending the isomorphism class of $\mathfrak{q}$ to the isomorphism class of $\mathfrak{q}^{\varpi_i}$. We are done.

A15. For any k^*-group $\hat{G}$, recall that we denote by $\mathcal{G}_k(\hat{G})$ the *scalar extensions* from $\mathbb{Z}$ to $\mathcal{O}$ of the Grothendieck group of the categories of finite-dimensional $k_*\hat{G}$-modules; it is well-known that we have a *contravariant* functor

$$\mathfrak{g}_k : k^*\text{-}\mathfrak{Gr} \longrightarrow \mathcal{O}\text{-}\mathfrak{mod} \qquad A15.1$$

mapping $\hat{G}$ on $\mathcal{G}_k(\hat{G})$ and any k^*-group homomorphism $\hat{\varphi} : \hat{G} \to \hat{G}'$ on the corresponding *restriction* map. Then, for any *folded Frobenius category* $(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$, we consider the composed functor

$$\mathfrak{ch}^*(\mathcal{F}^{^{\mathrm{sc}}}) \xrightarrow{\widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}}} k^*\text{-}\mathfrak{Gr} \xrightarrow{\mathfrak{g}_k} \mathcal{O}\text{-}\mathfrak{mod} \qquad A15.2$$

and we define the (modular) *Grothendieck group* of $(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ as the inverse limit

$$\mathcal{G}_k(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}}) = \lim_{\longleftarrow} (\mathfrak{g}_k \circ \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}}) \qquad A15.3;$$

at this point, it follows from [9, Corollary 8.4] suitably adapted and from Theorem A11 above that the $\mathbb{Z}$-valued function r mapping $(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})$ on $\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(P, \mathcal{F}, \widehat{\mathfrak{aut}}_{\mathcal{F}^{\mathrm{sc}}})\big)$ is a *radical function* and, if the Alperin Conjecture holds, it is easily checked from [1, Theorem 3.8] and from Theorem A11 above that r^* maps any k^**-localizer* $\hat{L}$ on the number of blocks of *defect zero* of the quotient $\hat{L}/\mathbb{O}_p(\hat{L})$.

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